Runge-Kutta Order 4 Approximation

1 Foundation

Before I go further, I want to provide some background and foundation information that will be useful later in this paper.

1.1 Taylor Series/Taylor Polynomials

$$f(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{f''(x_n)(x-x_n)^2}{2!} + \frac{f'''(x_n)(x-x_n)^3}{3!} + \frac{f^{(\text{iv})}(x_n)(x-x_n)^4}{4!} + \ldots + \frac{f^{(n)}(x_n)(x-x_n)^n}{n!} + \ldots + \frac{f^{(n+1)}(x_n)(x-x_n)^{n+1}}{(n+1)!} + \frac{f^{(n+2)}(x_n)(x-x_n)^{n+2}}{(n+2)!} + \ldots \quad (1)$$

The above series is known as the Taylor's Series. This series provides a way to approximate functions using polynomials. As you can see written in this form, the series is based heavily on the derivative of a continous function $\boldsymbol{f(x)}$.

$$\mathcal{P}(x) \approx f(x_n) + f'(x_n)(x - x_n) + \frac{f''(x_n)(x-x_n)^2}{2!} + \frac{f'''(x_n)(x-x_n)^3}{3!} + \frac{f^{(\text{iv})}(x_n)(x-x_n)^4}{4!} + \ldots + \frac{f^{(n)}(x_n)(x-x_n)^n}{n!} \quad (2)$$

$$\mathcal{R}(x) = \frac{f^{(n+1)}(x_n)(x-x_n)^{n+1}}{(n+1)!} + \frac{f^{(n+2)}(x_n)(x-x_n)^{n+2}}{(n+2)!} + \ldots \quad (3)$$

What I have done with (2) and (3) is to break the Taylor Series into two parts. (2) states that the partial sum provides an approximation for the function in question. (3) indicates the remainder can be calculated from the infinites series. Instead of representing the remainder as an infinite series, (3) can be represented in La Grange form as:

$$\mathcal{R}(x) = \frac{f^{(n+1)}(z)(x-z)^{n+1}}{(n+1)!} \text{ where } x < z < x_n \quad \text{or} \quad x_n < z < x \quad (4)$$

La Grange was a mathematician that proved that there exist a number z which exist between x and x_n such that the sum of the series (3) equals $\mathcal{R}(x)$.

1.2 Taylor Series Connection

The purpose for starting out discussing the Taylor Series is because many derivations of the Runge-Kutta methods base their parameters on the coefficients of the Taylor Series. In order to proceed further, it will be necessary to re-write (1) into a form that will help show the connection between the Taylor Series and the Runge-Kutta methods.

Let $\boldsymbol{h = x - x_n}$ **then** (1) can be re – written as such :

$$f(x) = f(x_n) + f'(x_n)(h) + \frac{f''(x_n)(h)^2}{2!} + \frac{f'''(x_n)(h)^3}{3!} + \frac{f^{(iv)}(x_n)(h)^4}{4!} + \ldots + \frac{f^{(n)}(x_n)(h)^n}{n!} + \ldots + \frac{f^{(n+1)}(x_n)(h)^{n+1}}{(n+1)!} + \frac{f^{(n+2)}(x_n)(h)^{n+2}}{(n+2)!} + \ldots \quad (5)$$

The difference between (1) and (5) is purely cosmetic but the next transformation of (1) helps provide clarity about the type of derivatives that are being discussed in this paper.

$$\mathcal{Y}' = f(x, y(x)) \quad (6)$$

As can be seen in (6), the first derivative is based on *two variables* and $\mathcal{Y}$ *is a function of* $\mathcal{X}$. It is important to draw attention to (6) because it will be necessary to use the *Chain-Rule* and *implicit differentation* to obtain the derivations needed to show the Taylor Series and Runge-Kutta connection.

$$\mathcal{Y}_{n+1} = \mathcal{Y}(x_n) + f(\mathcal{X}_n, \mathcal{Y}_n)(h) + \frac{f'(\mathcal{X}_n, \mathcal{Y}_n)(h)^2}{2!} + \frac{f''((\mathcal{X}_n, \mathcal{Y}_n)(h)^3}{3!} + \frac{f'''((\mathcal{X}_n, \mathcal{Y}_n)(h)^4}{4!} + \ldots + \frac{f^{(n-1)}((\mathcal{X}_n, \mathcal{Y}_n)(h)^n}{n!} + \ldots + \frac{f^{(n}(x_n)(h)^{n+1}}{(n+1)!} + \frac{f^{(n+1)}(x_n)(h)^{n+2}}{(n+2)!} + \ldots \quad (7)$$

(7) varies from (5) and (1) in that instead of focusing on deriving a polynomial approximation for a function $f(x)$, the focus is on how to calculate an approximation for $\mathcal{Y}_{n+1}$ based on values of $\mathcal{X}_n, \mathcal{Y}_n$. It is important to point out that $f(\mathcal{X}_n, \mathcal{Y}_n)$ in (7) represents the 1st derivative. When looking at (1) or (5) and comparing it to (7), it is important to note that $f(x_n)$ and $\mathcal{Y}(x_n)$ are equivalent.

1.3 Euler's Method

Euler's method was discussed in my previous paper and it is here for two reasons. The first is that the approximation technique will be compared to the Runge-Kutta 4 method. The second reason is that is serves as a reference for review as I show how to derive the Euler's method from the definition of Runge-Kutta.

To approximate the solution of an initial value problem using Euler's method perform the following steps:

$$\mathcal{Y}' = f(x, y), \quad \mathcal{Y}(x_0) = \mathcal{Y}_0 \quad (8)$$

Step 1. Choose a nonzero number h to serve as an increment or step size along the x-axis and let

$$\mathcal{X}_1 = \mathcal{X}_0 + h, \quad \mathcal{X}_2 = \mathcal{X}_1 + h, \quad \mathcal{X}_3 = \mathcal{X}_2 + h, \ \ldots \quad (9)$$

Step 2. Compute successively

$$\mathcal{Y}_1 = \mathcal{Y}_0 + f(\mathcal{X}_0, \mathcal{Y}_0)\,h \quad (10)$$

$$\mathcal{Y}_2 = \mathcal{Y}_1 + f(\mathcal{X}_1, \mathcal{Y}_1)\,h \quad (11)$$

$$\mathcal{Y}_3 = \mathcal{Y}_2 + f(\mathcal{X}_2, \mathcal{Y}_2)\,h \quad (12)$$

.

$$\mathcal{Y}_{n+1} = \mathcal{Y}_n + f(\mathcal{X}_n, \mathcal{Y}_n)\,h \quad (13)$$

The numbers $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots$ in the above equations are the approximations of $\mathcal{Y}(x_1), \mathcal{Y}(x_2), \mathcal{Y}(x_3), \ldots$.

(Source: Calculus A New Horizon: Author: Howard Anton)

1.4 Improved Euler's Method

There is an enhanced version of Euler's method known as the Improved Euler's method. The Improved Euler's method is included in this paper to serves as a reference for review and it will be used in a comparison of accuracy against Euler's method and Runge-Kutta 4 method. This method is defined as:

$$\mathcal{Y}_{n+1} = \mathcal{Y}_n + h\left(\frac{f(\mathcal{X}_n, \mathcal{Y}_n) + f(\mathcal{X}_{n+1}, \mathcal{Y}^*_{n+1})}{2}\right) \tag{14}$$

$$\mathcal{Y}^*_{n+1} = \mathcal{Y}_n + f(\mathcal{X}_n, \mathcal{Y}_n)\, h \tag{15}$$

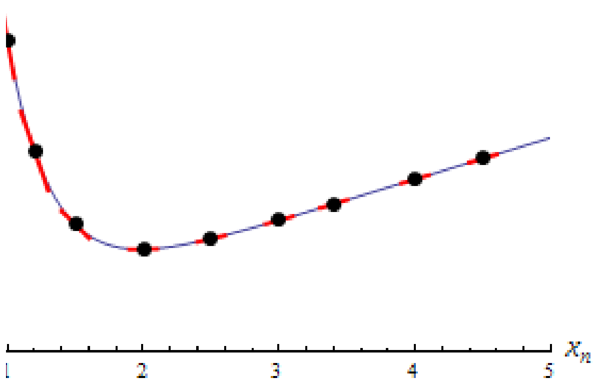
The value of $\mathcal{Y}^*_{n+1}$ given by (15) (the original form of Euler's method) predicts the value of $\mathcal{Y}(x_n)$. The value of $\mathcal{Y}_{n+1}$ defined by (14) corrects the estimate provided in (15)..

Source: Differential Equations with Boundary-Value Problems (Fifth Edition) Author(s): Dennis G. Zill and Michael R. Cullen.

2 Qualatative Approximation

2.1 Direction Fields

The slope interpretation of y' makes it possible to get a qualitative evaluation of a differential equation. Direction fields or slope fields present a set of integral curve solutions within a region. These graphs are created by calculating the slope of tangent lines at various points on each integral curve and then drawing only a small segment of the actual tangent line. The picture on the left side of **Figure 1.** contains the actual solution curve for **(32) .** The tangent line segments have been drawn on the actual solution curve to show how the tangent lines can be used to to get a graphical picture of how the curve behaves. The picture on the right side of **Figure 1.** does not contain the solution curve but it helps to emphasize how the tangent line segments can be used to see what the actual solution curve looks like. **Figures (2)** and **(3)** represent direction field diagrams for **(32)** and **(33)**



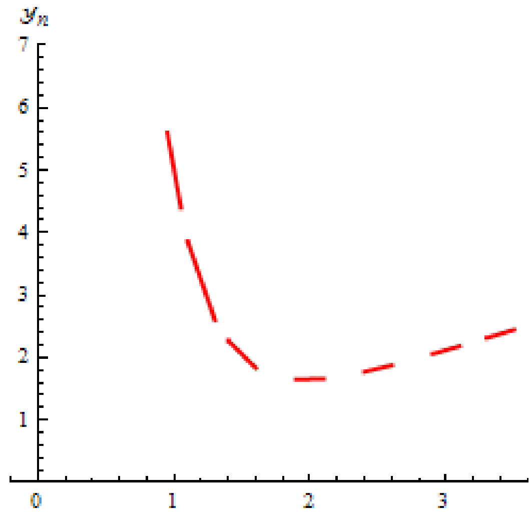


Figure 1. Graph of Particular Solution for Example 1

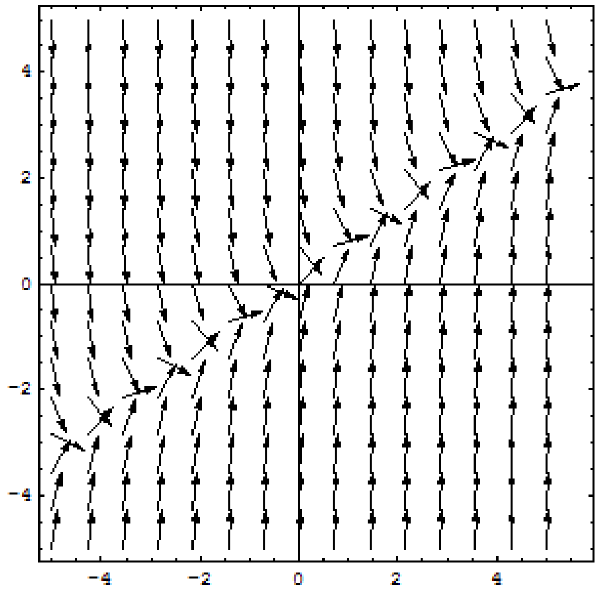


Figure 2. Direction Field for $y' = 2x - 3y + 1$

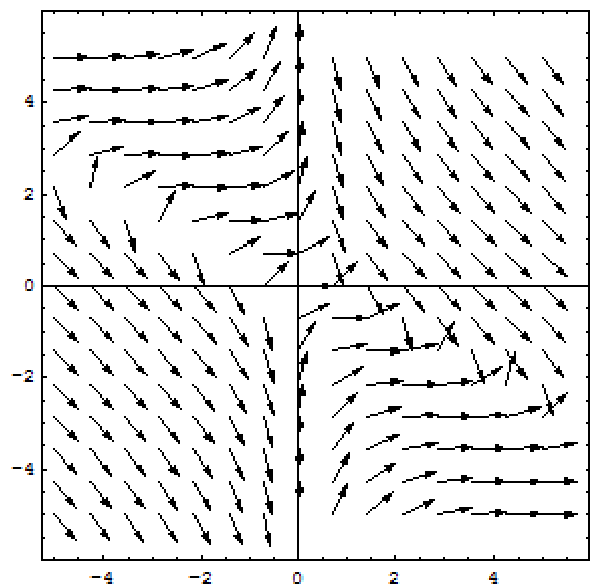


Figure 3. Direction Field for $y' = \frac{-(x+y)^2}{(2xy + x^2 - 1)}$

3 Runge-Kutta Methods

The Runge-Kutta methods are a group of approximation methods used on 1st order differential initial value problems of the type: $\boldsymbol{y' = f(x, y)}$, $\boldsymbol{y(x_0) = y_0}$. Runge-Kutta methods have different orders that are usually based on the degree of Taylor Polynomials. As in the case of Euler's method, Runge-Kutta methods interprets y' as the slope of the tangent line then; the differential equation states that at each point (x,y) on the integral curve, the slope of the tangent line $y' = f(x, y)$ is equal to the value of $y(x)$ at that point.

3.1 Runge-Kutta Definition

In order to approximate a function, the following is used:

$$\boldsymbol{y_{n+1} = y_n + h(w_1 k_1 + w_2 k_2 + w_3 k_3 + \ldots + w_m k_m)} \tag{16}$$

[Note : This slope function represents the weighted averages over the interval $\boldsymbol{X_n \leq X \leq X_{n+1}}$]

The weights w_i, i = 1,2,3, ...,m are constants that satisfy:

$$w_1 + w_2 + w_3 + \ldots + w_m = 1$$

Each k_i, i = 1,2,3,...,m is the function $\boldsymbol{f(x, y)}$ evaluated at a selected point (x, y) for which $X_n \leq X \leq X_{n+1}$. What is important to point out is that **m** represents the order of the Runge-Kutta method

Source:: A First Course in Differential Equations (With Applied Modeling) Eighth Edition. Authors: Dennis G. Zill.

3.1.1 Runge-Kutta Method One (RK1)

Manipulation of the parameters in (16) , so that **m** = 1, and $\boldsymbol{w_1} = 1$ creates a Runge-Kutta Method of Order One and (16) becomes

$$y_{n+1} = y_n + h(k_1) \tag{17}$$

$$\text{And since} \quad k_1 = f(X_n, y_n) = \frac{dy}{dx} \quad \text{then}$$

Equation (17) is the same as the equation (13) which establishes Euler's method to be the same as the 1st order Runge-Kutta Method.

Taylor Series/Taylor Polynomial

A Taylor Polynomial of degree 1 written in the context of (7) looks like:

$$y_{n+1} = y(x_n) + f(X_n, y_n)(h) + \frac{f'(X_n, y_n)(h)^2}{2!} \tag{18}$$

In order to derive a Taylor Polynomial of degree 1, two derivatives must be calculated with the last term being the remainder/error term. If the last term of (18) is omitted , a review of the coefficients in front of the first derivative shows that the choice of setting **m** = 1, and $\boldsymbol{w_1} = 1$ in (17) was not abirtrary but rather chosen to be in agreement with (18). As stated earlier, the coefficients of the Taylor Polynomial/Series often influence the parameters used to define the various Runge-Kutta methods.

3.1.2 Runge-Kutta Method Two (RK2)

The definition for the RK2 method is:

$$\mathcal{Y}_{n+1} = \mathcal{Y}_n + h\,(w_1\,k_1 + w_2\,k_2) \qquad \text{where} \tag{19}$$

$$k_1 = f(\mathcal{X}_n, \mathcal{Y}_n)$$
$$k_2 = f(\mathcal{X}_n + \text{h}\alpha, \mathcal{Y}_n + \text{h}\beta k_1)$$

Taylor Series/Taylor Polynomial Of One Variable

In order to begin the derivation of **RK2,** it will be necessary to derive a Taylor Series of Degree 2 (remember the last term is for error/remainder consideration).

$$\mathcal{Y}_{n+1} = \mathcal{Y}(x_n) + f(\mathcal{X}_n, \mathcal{Y}_n)\,(h) + \frac{f'(\mathcal{X}_n, \mathcal{Y}_n)\,(h)^2}{2!} + \frac{f''(x_n)\,(h)^3}{3!} \tag{20}$$

$$\mathcal{Y}_{n+1} = \mathcal{Y}(x_n) + \frac{dy}{dx}(\mathcal{X}_n, \mathcal{Y}_n) \times h + \frac{\frac{d^2 y}{dx^2}(\mathcal{X}_n, \mathcal{Y}_n)\,(h)^2}{2!} + \frac{\frac{d^3 y}{dx^3}(\mathcal{X}_n, \mathcal{Y}_n)\,(h)^3}{3!} \tag{21}$$

(20) and (21) are equivalent but they use a different notation. I will re-write the second and third derivatives of (21) in terms of partial derivatives. The key here is to remember that in order to have a first derivative of the form $f(\mathcal{X}, \mathcal{Y})$ means that we started with some function z = $g(\mathcal{X}, \mathcal{Y})$ and performed implicit differentiation using the Chain-Rule. The result of the implicit differentiation gave us another function of two variables.

The second derivative term of (21) written in terms of partials is:

$$\left[\frac{d}{dx}\left(\frac{dy}{dx}(\mathcal{X}_n, \mathcal{Y}_n)\right)\right] \times \frac{h^2}{2!} = \left[\frac{\partial f(\mathcal{X}_n, \mathcal{Y}_n)}{\partial x} + \frac{\partial f(\mathcal{X}_n, \mathcal{Y}_n)}{\partial y} \times \frac{dy}{dx}(\mathcal{X}_n, \mathcal{Y}_n)\right] \times \frac{h^2}{2!} = [y''] \times \frac{h^2}{2!} \tag{22}$$

The error/remainder term for (21) is:

$$\begin{aligned}\left[\frac{d}{dx}\left(\frac{d^2}{dx^2}\right)\right] \times \frac{h^3}{3!} = \Bigg[& \frac{\partial^2 f}{\partial x^2}(\mathcal{X}_n, \mathcal{Y}_n) + \\ & \frac{\partial^2 f}{\partial y\,\partial x}(\mathcal{X}_n, \mathcal{Y}_n) \times f(\mathcal{X}_n, \mathcal{Y}_n) + \frac{\partial^2 f}{\partial x\,\partial y}(\mathcal{X}_n, \mathcal{Y}_n) \times f(\mathcal{X}_n, \mathcal{Y}_n) + \\ & \frac{\partial^2 f}{\partial y^2}(\mathcal{X}_n, \mathcal{Y}_n) \times f(\mathcal{X}_n, \mathcal{Y}_n) \times f(\mathcal{X}_n, \mathcal{Y}_n) + \\ & \frac{\partial f}{\partial y}(\mathcal{X}_n, \mathcal{Y}_n) \times \left(\frac{\partial f}{\partial x}(\mathcal{X}_n, \mathcal{Y}_n) + \frac{\partial f}{\partial y}(\mathcal{X}_n, \mathcal{Y}_n) \times f(\mathcal{X}_n, \mathcal{Y}_n)\right)\Bigg] \frac{h^3}{3!} = [y'''] \times \frac{h^3}{3!}\end{aligned} \tag{23}$$

Taylor Series/Polynomial of Multiple Variables

Up until this point, it has been useful to work with Taylor Series/Polynomials of one variable . One had to remember that we are working with a function like $\mathcal{Y}' = f(x, y(x))$, where it was possible to apply implicit differentiation and the Chain-Rule to obtain the derivatives that were needed. To better understand what is going on with the derivation of the RK2 method, it will be useful to apply the multivariate version of Taylor Polynomial/Series. The multivariate version is defined as :

$$f(x_1, \ldots, x_n) = \sum_{j=0}^{\infty}\left\{\frac{1}{j!}\left(\sum_{k=1}^{\infty}(x_k + a_k)\frac{\partial}{\partial x_k^j}\right)^j f(x'_1, \ldots, x'_n)\right\} \tag{24}$$

Source: Taylor Series - Wolfram Mathematica

(19) provides the definition of the RK2 method. There are two parts of the definition that will need to be made to conform to the Taylor Polynomial of two degrees. The first is:

$$h\,(w_1\,k_1)$$

This is the first derivative multiplied by the step size **(h)** and a constant $\boldsymbol{w_1}$. Using the coefficients of the first derivative in (21) as a guide, yields that $\boldsymbol{w_1}$ =1 is a good choice and this gives

$$\frac{dy}{dx}\,(\mathcal{X}_n,\,\mathcal{Y}_n)\times h\times 1 \;=\; h\,(w_1\,k_1) \tag{25}$$

The second part of (19) that will need to be made to conform to the Taylor Polynomial is::

$$h\,(w_2\,k_2), \qquad \text{where} \qquad k_2 \;=f(\mathcal{X}_n\;+\mathrm{h}\alpha,\,\mathcal{Y}_n\;+\;\mathrm{h}\beta k_1)$$

Using (24) to help see how to construct $\boldsymbol{k_2}$, yeilds:

$$f(\mathcal{X}_n\;+\mathrm{h}\alpha,\,\mathcal{Y}_n\;+\;\mathrm{h}\beta k_1)\;=\;f(\mathcal{X}_n,\,\mathcal{Y}_n)\;+[\;\frac{\partial f}{\partial x}\,(\mathcal{X}_n,\,\mathcal{Y}_n)\times\alpha h\;+\;\frac{\partial f}{\partial y}\,(\mathcal{X}_n,\,\mathcal{Y}_n)\,\times\beta h\times f(\mathcal{X}_n,\,\mathcal{Y}_n)]\;+ \tag{26}$$

$$\frac{1}{2!}\Bigg[$$

$$(\alpha\times h)^2\times\frac{\partial^2 f}{\partial x^2}\,(\mathcal{X}_n,\,\mathcal{Y}_n)\;+\;\;2\,\alpha h^2\,\beta\times f(\mathcal{X}_n,\,\mathcal{Y}_n)\times\frac{\partial^2 f}{\partial xy}\,(\mathcal{X}_n,\,\mathcal{Y}_n)\;\;+\;(\beta h\,f(\mathcal{X}_n,\,\mathcal{Y}_n))^2\times\;\frac{\partial^2 f}{\partial y^2}\,(\mathcal{X}_n,\,\mathcal{Y}_n)\Bigg]$$

With (26) it is now possible to start seeing how the RK2 method is derived. First (20) will be re-written to use the partial form of the second derivative as shown in (22) and we will ignore the error/remainder term.. This change to (20) looks like:

$$\mathcal{Y}_{n+1}\;\;=\;\;\mathcal{Y}(x_n)\;+\;f(\mathcal{X}_n,\,\mathcal{Y}_n)\,(h)\;+\!\left[\frac{\partial f(\mathcal{X}_n,\,\mathcal{Y}_n)}{\partial x}+\frac{\partial f(\mathcal{X}_n,\,\mathcal{Y}_n)}{\partial y}\times\frac{dy}{dx}\,(\mathcal{X}_n,\,\mathcal{Y}_n)\right]\times\frac{h^2}{2!} \tag{27}$$

Now I will start to expand the definition provided in (19) using (25) and (26) , this gives the following:

$$\begin{gathered}\mathcal{Y}_{n+1}\;\;=\mathcal{Y}_n\;+\;f(\mathcal{X}_n,\,\mathcal{Y}_n)\times(h)\times\,(w_1)\;+\\ (h)\times\,(w_2)\times\boldsymbol{[}\,f(\mathcal{X}_n,\,\mathcal{Y}_n)\;+[\;\frac{\partial f}{\partial x}\,(\mathcal{X}_n,\,\mathcal{Y}_n)\times\alpha h\;+\;\frac{\partial f}{\partial y}\,(\mathcal{X}_n,\,\mathcal{Y}_n)\,\times\beta h\times f(\mathcal{X}_n,\,\mathcal{Y}_n)]\boldsymbol{]}\end{gathered} \tag{28}$$

In the next steps, I will start grouping terms so that it will be easier to see how the parameters $\boldsymbol{w_1}$,$\boldsymbol{w_2}$,$\boldsymbol{\alpha}$,and $\boldsymbol{\beta}$ are choosen:

$$\begin{gathered}\mathcal{Y}_{n+1}\;\;=\mathcal{Y}_n\;\;+[\,f(\mathcal{X}_n,\,\mathcal{Y}_n)\times(h)\times\,(w_1)]\;\;+\,(h)\times\,(w_2)\times\,f(\mathcal{X}_n,\,\mathcal{Y}_n)\;\;+\\ \frac{\partial f}{\partial x}\,(\mathcal{X}_n,\,\mathcal{Y}_n)\times\alpha h^2\,w_2\;+\;\;\frac{\partial f}{\partial y}\,(\mathcal{X}_n,\,\mathcal{Y}_n)\;(w_2)\,\beta h^2\times f(\mathcal{X}_n,\,\mathcal{Y}_n)\end{gathered} \tag{29}$$

$$\mathcal{Y}_{n+1}\;\;=\mathcal{Y}_n\;\;+\;\;f(\mathcal{X}_n,\,\mathcal{Y}_n)\times(h)[\,(w_1+w_2)]\;+\;h^2\,w_2\!\left[\frac{\partial f}{\partial x}\,(\mathcal{X}_n,\,\mathcal{Y}_n)\times\alpha\;+\;\frac{\partial f}{\partial y}\,(\mathcal{X}_n,\,\mathcal{Y}_n)\,\times\beta\right] \tag{30}$$

Look very close at (27) which is a Taylor Polynomial and (30) which is the definition of RK2, the first derivative of (27) has the coefficients $\boldsymbol{h}$ and **1.** In (30) the coefficients of the first derivative are **h** and $\boldsymbol{(w_1+w_2)}$, This suggests that a choice of $\boldsymbol{w_2}\;=\;\frac{1}{2}$, $\boldsymbol{w_1}\;=\;\frac{1}{2}$ will get the first derivative of (30) to match the first derivative of (27). Now look at the coefficients of the second derivative of (27) and compare that with the coefficients of the second derivative of (30) since a choice of $w_2 = \frac{1}{2}$ was made then setting $\alpha = 1$, $\boldsymbol{\beta} = \times 1$ will put the second derivative of (30) in agreement with (27).

As it has been shown, the RK2 method can be influenced by the Taylor Polynomial/Series but it must be pointed out that the choices for the parameters $\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{\alpha}$, and $\boldsymbol{\beta}$ could have been made differently and still satisfied (19). This means that there are many different forms of RK2.

3.1.3 Runge-Kutta Method Four (RK4)

In this paper, the RK4 method will not be derived but the approach would be similar to what has been shown for the RK1 and RK2 methods. The definition of the RK4 Method is:

$$\mathcal{Y}_{n+1} = \mathcal{Y}_n + h\,(w_1\,k_1 + w_2\,k_2 + w_3\,k_3 + w_4\,k_4) \text{ where} \tag{31}$$

$$\begin{aligned} k_1 &= f(\mathcal{X}_n, \mathcal{Y}_n) \\ k_2 &= f(\mathcal{X}_n + \mathrm{h}\alpha_1, \mathcal{Y}_n + \mathrm{h}\beta k_1) \\ k_3 &= f(\mathcal{X}_n + \mathrm{h}\alpha_2, \mathcal{Y}_n + \mathrm{h}\beta_2\,k_1 + \mathrm{h}\beta_3\,k_2) \\ k_4 &= f(\mathcal{X}_n + \mathrm{h}\alpha_3, \mathcal{Y}_n + \mathrm{h}\beta_4\,k_1 + \mathrm{h}\beta_5\,k_2 + \mathrm{h}\beta_6\,k_3) \end{aligned}$$

Source:: A First Course in Differential Equations (With Applied Modeling) Eighth Edition. Authors: Dennis G. Zill.

4 Runge-Kutta Method 4 Example and Comparison

4.1 Example #1

The example is based on the problem done in the first paper which was used to compare Euler's method and the Improved Euler's method. I will provide the background information so that you can see the solution

4.1.1 Special Note

One way to improve the accuracy of the approximation techniques mentioned in this paper is to decrease the **step size (*h*).** For the sake of consistency, I have used the same step size so that an adequate comparison can be made. There is one exception and that is with **Table 3.** and **Figure 5.** I chose a smaller step size than the one used for the RK4 method to demonstrate the superiority of the RK4 method over Euler's method.

$$\begin{aligned} \mathcal{Y}' &= 2\,x - 3\,y + 1 \\ d\,y/d\,x + 3\,y &= 2\,x + 1 \\ \mu &= e^{\int p(x)\,dx} \\ \mu &= e^{\int 3\,dx} \\ \mu &= e^{3\,x} \qquad \text{(The Integrating Factor)} \\ d\,/d\,x\left[e^{3\,x}\,y\right] &= e^{3\,x}(2\,x+1)\,d\,x \\ \int d\,/d\,x\left[e^{3\,x}\right] &= \int e^{3\,x}(2\,x+1)\,d\,x \\ \int u\,d\,v &= \mathrm{uv} - \int v\,d\,u \;\; \text{(Integration By Parts Rule)} \\ \text{let } u &= 2\,x+1, \text{ then du } = 2\,\mathrm{dx} \\ \text{let dv} &= e^{3\,x}, \text{ then } v = \int d\,v = \int e^{3\,x}\,d\,x = \frac{e^{3\,x}}{3} \end{aligned} \tag{32}$$

$$\int e^{3x}(2x+1)\,dx = \frac{e^{3x}(2x+1)}{3} - \frac{2}{3}\int e^{3x}\,dx$$

$$\int e^{3x}(2x+1)\,dx = \frac{e^{3x}(2x+1)}{3} - \frac{2e^{3x}}{9} + c$$

$$e^{3x}y = \frac{e^{3x}(2x+1)}{3} - \frac{2e^{3x}}{9} + c$$

$$y = \frac{2x}{3} + \frac{1}{9} + \frac{c}{e^{3x}} \quad \text{(General Solution)}$$

$$\mathcal{Y}(1) = 5$$

$$5 = \frac{2}{3} + \frac{1}{9} + \frac{c}{e^{3x}}$$

$$\frac{38e^3}{9} = c$$

$$y = \frac{2x}{3} + \frac{1}{9} + \frac{38e^3}{9e^{3x}} \quad \text{(Particular Solution)}$$

Table 1. Results from the Various Approximation Techniques **Note: h = 0.1** **Exanple #1**

$\mathcal{X}_n$	*Actual $\mathcal{Y}_n$*	*Euler's Method $\mathcal{Y}_n$*	*Improved Euler's $\mathcal{Y}_n$*	*Runge-Kutter 4 $\mathcal{Y}_n$*
$\mathcal{X}_0 = 1.0$	$\mathcal{Y}_0 = 5.0$	$\mathcal{Y}_0 = 5.0$	$\mathcal{Y}_0 = 5.0$	$\mathcal{Y}_0 = 5.0$
$\mathcal{X}_1 = 1.1$	$\mathcal{Y}_1 = 3.97234$	$\mathcal{Y}_1 = 3.8$	$\mathcal{Y}_1 = 3.99$	$\mathcal{Y}_1 = 3.97245$
$\mathcal{X}_2 = 1.2$	$\mathcal{Y}_2 = 3.22832$	$\mathcal{Y}_2 = 2.98$	$\mathcal{Y}_2 = 3.25455$	$\mathcal{Y}_2 = 3.22844$
$\mathcal{X}_3 = 1.3$	$\mathcal{Y}_3 = 2.69441$	$\mathcal{Y}_3 = 2.426$	$\mathcal{Y}_3 = 2.72364$	$\mathcal{Y}_3 = 2.69454$
$\mathcal{X}_4 = 1.4$	$\mathcal{Y}_4 = 2.31615$	$\mathcal{Y}_4 = 2.0582$	$\mathcal{Y}_4 = 2.34511$	$\mathcal{Y}_4 = 2.31639$
$\mathcal{X}_5 = 1.5$	$\mathcal{Y}_5 = 2.05322$	$\mathcal{Y}_5 = 1.82074$	$\mathcal{Y}_5 = 2.08011$	$\mathcal{Y}_5 = 2.05334$

Table 2. Analysis of Euler's Method Approximation **Note: h = 0.1**

X_n	Euler's Method y_n	Actual y_n	Absolute Error	Percentage Relative Error (%)
$X_0 = 1.0$	$y_0 = 5.0$	$y_0 = 5.0$	0.0	0.0
$X_1 = 1.1$	$y_1 = 3.8$	$y_1 = 3.97234$	0.17234	4.3385
$X_2 = 1.2$	$y_2 = 2.98$	$y_2 = 3.22832$	0.24832	7.69193
$X_3 = 1.3$	$y_3 = 2.426$	$y_3 = 2.69441$	0.26841	9.96174
$X_4 = 1.4$	$y_4 = 2.0582$	$y_4 = 2.31615$	0.25795	11.137
$X_5 = 1.5$	$y_5 = 1.82074$	$y_5 = 2.05322$	0.23248	11.3227

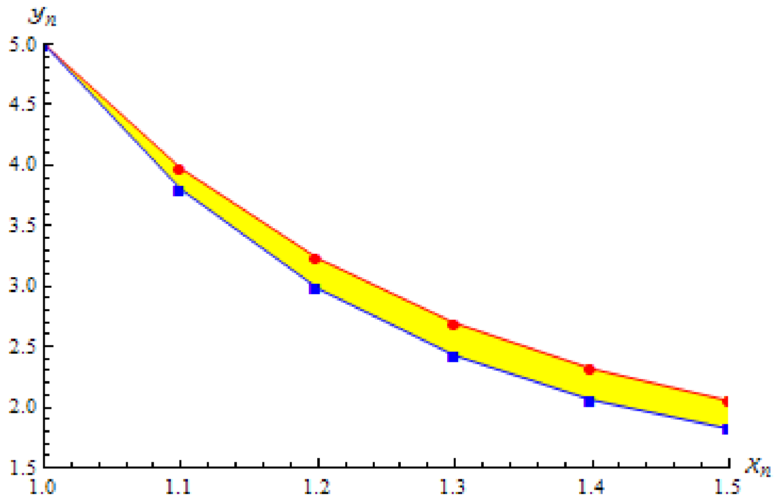


Figure 4. The Red curve represents y_n *(Actual)* The Blue curve represents y_n *(Approximation Euler Method)*

Table 3. Analysis of Euler's Method Approximation. **Note: h = 0.05**

X_n	*Euler's Method* y_n	*Actual* y_n	*Absolute Error*	*Percentage Relative Error (%)*
$X_0 = 1.00$	$y_0 = 5.0$	$y_0 = 5.0$	0.0	0.0
$X_1 = 1.05$	$y_1 = 4.400$	$y_1 = 4.4452$	0.045211	1.01708
$X_2 = 1.10$	$y_2 = 3.895$	$y_2 = 3.97234$	0.77343	1.94705
$X_3 = 1.15$	$y_3 = 3.471$	$y_3 = 3.56998$	0.099235	2.7797
$X_4 = 1.20$	$y_4 = 3.1151$	$y_4 = 3.22832$	0.11317	3.50580
$X_5 = 1.25$	$y_5 = 2.8718$	$y_5 = 2.93888$	0.12101	4.11769
$X_6 = 1.30$	$y_6 = 2.5701$	$y_6 = 2.69440$	0.12421	4.61023
$X_7 = 1.35$	$y_7 = 2.3646$	$y_7 = 2.48862$	0.12396	4.98135
$X_8 = 1.40$	$y_8 = 2.1949$	$y_8 = 2.31615$	0.12119	5.23252
$X_9 = 1.45$	$y_9 = 2.0557$	$y_9 = 2.17234$	0.116631	5.36892
$X_{10} = 1.50$	$y_{10} = 1.9423$	$y_{10} = 2.05322$	0.110857	5.39921

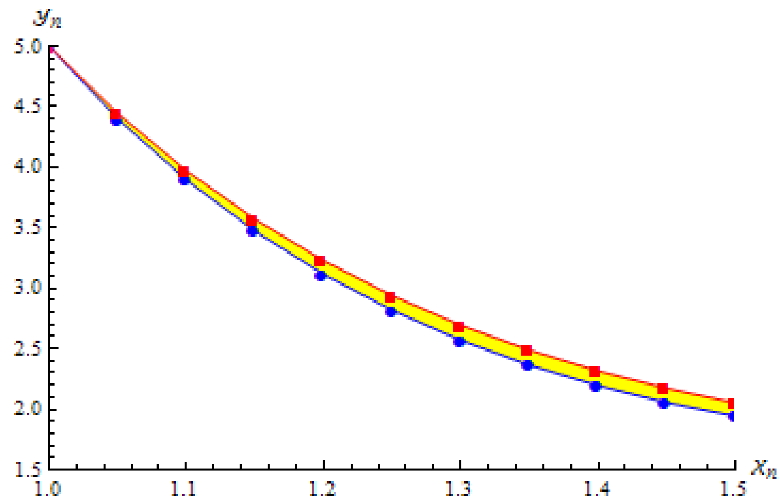


Figure 5. The Red curve represents y_n *(Actual)* The Blue curve represents y_n *(Approximation Euler Method)*

Table 4. Analysis of Improved Euler's Method Approximation **Note: h = 0.1**

X_n	Improved Euler's Method Y_n	Actual Y_n	Absolute Error	Percentage Relative Error (%)
$X_0 = 1.0$	$Y_0 = 5.0$	$Y_0 = 5.0$	0.0	0.0
$X_1 = 1.1$	$Y_1 = 3.99$	$Y_1 = 3.97234$	0.01766	0.444843
$X_2 = 1.2$	$Y_2 = 3.25455$	$Y_2 = 3.22832$	0.02623	0.812628
$X_3 = 1.3$	$Y_3 = 2.72364$	$Y_3 = 2.69441$	0.02923	1.08500
$X_4 = 1.4$	$Y_4 = 2.34511$	$Y_4 = 2.31615$	0.02896	1.25027
$X_5 = 1.5$	$Y_5 = 2.08011$	$Y_5 = 2.05322$	0.02689	1.30974

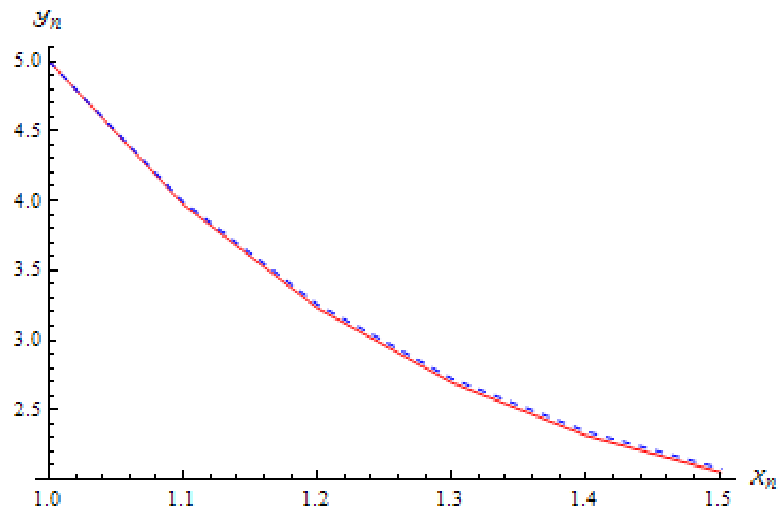


Figure 6. The Red curve represents Y_n *(Actual)* The Blue Dashed curve represents Y_n *(Approximation Improved Euler Method)*.

Table 5. Analysis of Runge-Kutta 4 Method Approximation **Note h = 0.1**

$\mathcal{X}_n$	*RK4 Method* $\mathcal{Y}_n$	*Actual* $\mathcal{Y}_n$	*Absolute Error*	*Percentage Relative Error (%)*
$\mathcal{X}_0 = 1.0$	$\mathcal{Y}_0 = 5.0$	$\mathcal{Y}_0 = 5.0$	0.0	0.00
$\mathcal{X}_1 = 1.1$	$\mathcal{Y}_1 = 3.97243$	$\mathcal{Y}_1 = 3.97234$	0.00009	0.002266
$\mathcal{X}_2 = 1.2$	$\mathcal{Y}_2 = 3.22844$	$\mathcal{Y}_2 = 3.22832$	0.00012	0.003717
$\mathcal{X}_3 = 1.3$	$\mathcal{Y}_3 = 2.69454$	$\mathcal{Y}_3 = 2.69441$	0.00013	0.004825
$\mathcal{X}_4 = 1.4$	$\mathcal{Y}_4 = 2.31629$	$\mathcal{Y}_4 = 2.31615$	0.00014	0.006045
$\mathcal{X}_5 = 1.5$	$\mathcal{Y}_5 = 2.05334$	$\mathcal{Y}_5 = 2.05322$	0.00012	0.005844

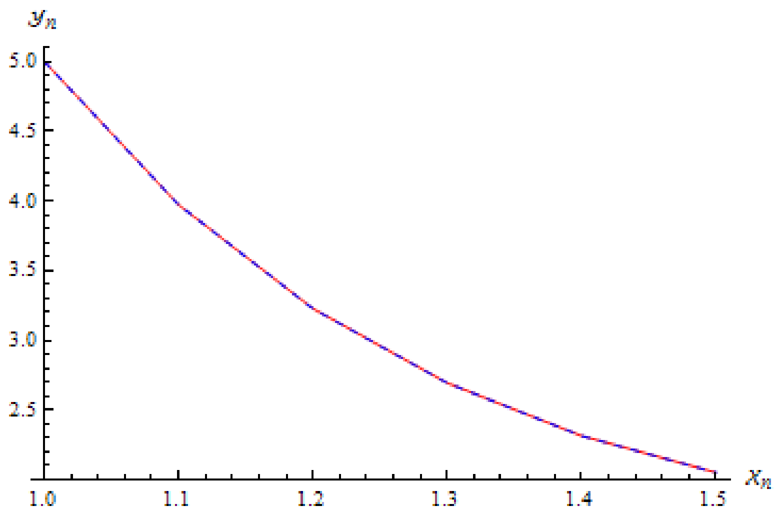


Figure 4. The Red curve represents - *(Actual)* The Blue Dashed curve represents $\mathcal{Y}_n$ *(Approximation RK4 Method)*

4.2 Example #2

This is an additional example comparing the various approximation techniques discussed in this paper.

$$(x+y)^2\,dx + \left(2\,xy + x^2 - 1\right)dy = 0 \quad , \quad y(1) = 1 \qquad (33)$$

$$dy\left[2\,xy + x^2 - 1\right] = -(x+y)^2\,dx$$

$$\frac{dy}{dx}\left[2\,xy + x^2 - 1\right] = -(x+y)^2$$

$$\frac{dy}{dx} = \frac{-(x+y))^2}{[2\,xy + x^2 - 1]}$$

A first order differential equation of the form $m(x,y)dx + n(x,y)dy = 0$ is exact if:

$$\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$$

Here a test for exactness takes place:

$$m = (x+y)^2$$

$$n = \left[2\,xy + x^2 - 1\right]$$

$$\frac{\partial m}{\partial y} = 2\,(x+y)$$

$$\frac{\partial n}{\partial x} = 2\,(x + y)$$

Proceed to find General and Particular Solution:

$$\frac{\partial \mathcal{F}}{\partial x} = (x + y)^2$$
$$\mathcal{F}(x,\, y) = \int (x + y)^2\, d\,x$$

Using substitution:

$$u = (x + y)^2$$
$$du = dx$$
$$\int u^2\, d\,u \; = \; \frac{u^3}{3} + c1$$

Translate back into original variables:

$$\int (x + y)^2\, d\,x = \frac{(x + y)^3}{3} + g(y)$$
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{(x + y)^3}{3} \; + \; g(y)\right]$$
$$\frac{\partial f}{\partial y} = (x + y)^2 \; + \; g'(y) \; = \; \left(2\,xy \; + \; x^2 \; - 1\right)$$
$$g'(y) = \left(2\,xy \; + \; x^2 \; - 1\right) - (x + y)^2$$
$$g'(y) = 2\,xy \; + \; x^2 \; - 1 \; - \; x^2 \; - 2\,xy \; - y^2$$
$$g'(y) = -1 \; - y^2$$
$$\int g'(y)\, d\,y = \int -1 \; - y^2\, d\,y$$
$$g(y) = \frac{-y^3}{3} - y \; + \; c$$
$$\mathcal{F}(x,\, y) = \frac{(x + y)^3}{3} \; \frac{-y^3}{3} - y \; + \; c \;\; \textbf{(General Solution)}$$
$$1 = \frac{8}{3} - \; \frac{1}{3} - 1 \; + c$$
$$2 \; - \; \frac{8}{3} \; + \; \frac{1}{3} = c$$
$$\frac{-4}{3} = c$$
$$\frac{4}{3} = \frac{x^3}{3} + \; x^2\, y \; + xy^2 - y \;\; \textbf{(Particular Solution)}$$

Table 6. Results from the Various Approximation Techniques **Note: h = 0.1** **Exanple #2**

$\mathcal{X}_n$	*Euler's Method* $\mathcal{Y}_n$	*Improved Euler's Method* $\mathcal{Y}_n$	*RK4 Method* $\mathcal{Y}_n$
$\mathcal{X}_0 = 1.0$	$\mathcal{Y}_0 = 1.0000$	$\mathcal{Y}_0 = 1.0000$	$\mathcal{Y}_0 = 1.0000$
$\mathcal{X}_1 = 1.1$	$\mathcal{Y}_1 = 0.8000$	$\mathcal{Y}_1 = 0.8083$	$\mathcal{Y}_1 = 0.8089$
$\mathcal{X}_2 = 1.2$	$\mathcal{Y}_2 = 0.6167$	$\mathcal{Y}_2 = 0.6309$	$\mathcal{Y}_2 = 0.6319$
$\mathcal{X}_3 = 1.3$	$\mathcal{Y}_3 = 0.4448$	$\mathcal{Y}_3 = 0.4630$	$\mathcal{Y}_3 = 0.4645$
$\mathcal{X}_4 = 1.4$	$\mathcal{Y}_4 = 0.2799$	$\mathcal{Y}_4 = 0.3006$	$\mathcal{Y}_4 = 0.3025$
$\mathcal{X}_5 = 1.5$	$\mathcal{Y}_5 = 0.1181$	$\mathcal{Y}_5 = 0.1399$	$\mathcal{Y}_5 = 0.1423$