Differential Techniques for 1st order ODE's

Introduction

For this topic, I have intentionally tried to minimize the amount of theory normally involved (definitions, theorems, proofs, etc) and simply focus on the mechanics of solving various types of first order differential equations.

Section

Background

A first order linear differential equation is defined as

$a_1(x)\,\frac{dy}{dx} + a_0(x)\,y = g(x)$. If (1) is multiplied by $\frac{1}{a_1(x)}$ as done in (2) (1)

$$\frac{a_1(x)}{a_1(x)}\,\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}\,y = \frac{g(x).}{a_1(x)} \tag{2}$$

Let $p(x) = \frac{a_0(x)}{a_1(x)}$ and let $f(\mathrm{x}) = \frac{g(x).}{a_1(x)}$ then (2) can be written in what is called the standard form (3)

$$\frac{dy}{dx} + p(x)\,y = f(x) \tag{3}$$

Separation of Variables

A differential equation that can be put into a form like (4) can be solved using a technique known as separation of variables. The **first step** requires analysis of the differential equation to determine if the method can be used. This is done by checking to see if one of the following is true:

(a) The differential equation is already in the desired form.

(b) The differential equation can be transformed into the desired form by using either algebraic manipulation and/or substitution.

The **second step** involves integration with respect to the appropriate variable as shown in (6). (5) is provided because sometimes after algebraic manipulation this is the end result. With (5) both sides require integration with respect to x but because $dy = y' \times dx$, (5) can be re-written as (6) and now the left side can be integrated with respect to y and the right side can be integrated with respect to x. Note: dy and dx are differentials.

The **third step** is done whenever substitution is performed. This step simply requires reverse substitution to put the answer back in terms of the original variables used in the problem.

$$g(y)\,y' = f(x) \tag{4}$$

$$\int g(y)\,y'\,dx = \int f(x)\,dx + c \tag{5}$$

$$\int g(y)\,dy = \int f(x)\,dx + c \tag{6}$$

Example requiring algebraic manipulation

$$y' = 2\sec(2\,y) \tag{7a}$$

$$\frac{y'}{\sec(2\,y)} = 2 \tag{7b}$$

$$\int g(y)\,y'\,dy = \int 2\,dx \tag{7c}$$

$$\int \frac{1}{\sec(2\,y)}\,dy = \int 2\,dx \tag{7d}$$

$$\int \cos(2\,y)\,dy = \int 2\,dx \tag{7e}$$

$$u = 2\,y \tag{7f}$$

$$du = 2\,dy \tag{7g}$$

$$\frac{du}{2} = dy \tag{7h}$$

$$\frac{1}{2}\int \cos u\,du = \int 2\,dx \tag{7i}$$

$$\frac{1}{2}[\mathrm{sinu} + c_2] = 2\,x + c_1 \tag{7j}$$

$$\sin(u) + c_2 = 4\,x + 2\,c_1 \tag{7k}$$

$$\sin(2\,y) + c_2 = 4\,x + 2\,c_1 \tag{7l}$$

$$\sin(2\,y) = 4\,x + 2\,c_1 - c_2 \tag{7m}$$

$$\sin(2\,y) = 4\,x + c \tag{7n}$$

$$\arcsin[\sin(2\,y)] = \arcsin[4\,x + c] \tag{7o}$$

$$2\,y = \arcsin[4\,x + c] \tag{7p}$$

$$y = \frac{\arcsin[4\,x + c]}{2} \tag{7q}$$

In the example above, Analyzing (7a) shows that the form is not correct. (7b) is the correct change needed. Substitution was used as shown in (7f) - (7i) but not to transform the equation into one that could be solved by the method of separation of variables but rather to help make the integration process easier to understand.

Reduction to Separable form

As mentioned earlier, substitution can be used to transform an ode from non-separable form to separable form. Ode's like (8) and (9) are two special cases that require some discussion.

$$y' = f\left(\frac{y}{x}\right) \tag{8}$$

Where f is a differentiable function such as : $\frac{y}{x}$, $\tan\left(\frac{y}{x}\right)$, $\left(\frac{y}{x}\right)^2$, etc.

$$y' = f(\mathcal{A}x + \mathcal{B}y + c) \tag{9}$$

Where f is a differentiable function such as $(\mathcal{A}x + \mathcal{B}y + c)^6$, $\cos(\mathcal{A}x + \mathcal{B}y + c)$, etc.

(10a) thru (10h) provide the details on how to use substitution for a differential equation like (8). The most important thing to understand is that the appropriate substitution is shown in (10a). The variables u and y are functions of x. This is important when performing the differentiation shown in (10c) which requires the usage of the chain rule. As you review the process, it is clear that by the time we reach (10h) everything on the left side is in terms of x and everything on the right side is in terms of u (the variables have been separated).

$$u = \frac{y}{x} \tag{10a}$$

$$y = ux \tag{10b}$$

$$y' = u' x + u \tag{10c}$$

$$f(u) = u' x + u \tag{10d}$$

$$f(u) = \left(\frac{du}{dx}\right) + u \tag{10e}$$

$$f(u) - u = \left(\frac{du}{dx}\right)x \tag{10f}$$

$$(f(u) - u)dx = x * du \tag{10g}$$

$$\frac{dx}{x} = \frac{du}{(f(u) - u)} \tag{10h}$$

In the following example, I attempt to transform (11a) by means of algebraic manipulation but in the end I am left with an equation that is not separable. By applying the substitution of (10a) and working the through process detailed in (10b) through (10h) I end up at (11g) which gives me a form that is appropriate for the method of separation of variables. As mention previously, the second general step is to perform integration with respect to the appropriate variables which is done in step (11h) -(11i). The final step of reverse substitution is shown on line (11n).

Example

$$y' = \left(4x^2 + y^2\right) / (xy) \tag{11a}$$

$$y' = \frac{4x^2}{xy} + \frac{y^2}{xy} \tag{11b}$$

$$y' = \frac{4x}{y} + \frac{y}{x} \tag{11c}$$

$$u' x + u = \frac{4}{u} + u \tag{11d}$$

$$u' x = \frac{4}{u} + u - u \tag{11e}$$

$$\frac{du}{dx} x = \frac{4}{u} \tag{11f}$$

$$\frac{u \times du}{4} = \frac{dx}{x} \tag{11g}$$

$$\int \frac{u}{4} du = \int \frac{1}{x} dx \tag{11h}$$

$$\left(\frac{1}{4}\right)\frac{u^2}{2} + c_1 = \ln(x) + c_2 \tag{11i}$$

$$\frac{1}{8} u^2 = \ln(x) + (c_2 - c_1) \tag{11j}$$

$$\frac{1}{8} u^2 = \ln(x) + c_3 \tag{11k}$$

$$u^2 = 8\,(\ln(x) + c_3) \tag{11l}$$

$$u^2 = 8\ln(x) + c \tag{11m}$$
$$\frac{y^2}{x^2} = 8\ln(x) + c \tag{11n}$$
$$y^2 = (x^2)(8\ln(x) + c) \tag{11o}$$

The solution is

$y = x\sqrt{\ln(x) + c}$ and

$y = -x\sqrt{\ln(x) + c}$

For ode's like (9), the appropriate substitution is shown in (12a). (12b) is obtained by differentiating with respect to x and applying the chain rule as required. $\mathcal{A}$ and $\mathcal{B}$ are constant coefficients.

$$u = \mathcal{A}x + \mathcal{B}y + c \tag{12a}$$
$$\frac{du}{dx} = \mathcal{A} + \mathcal{B}\frac{dy}{dx} \tag{12b}$$
$$\left(\frac{du}{dx} - \mathcal{A}\right)\frac{1}{\mathcal{B}} = \frac{dy}{dx} \tag{12c}$$

Example

$$y' = (x + y + 1)^2 \tag{13a}$$
$$u = x + y + 1 \tag{13b}$$
$$\frac{du}{dx} = 1 + \frac{dy}{dx} \tag{13c}$$
$$\frac{dy}{dx} = \frac{du}{dx} - 1 \tag{13d}$$
$$\frac{du}{dx} - 1 = u^2 \tag{13e}$$
$$\frac{du}{dx} = u^2 + 1 \tag{13f}$$
$$du = (u^2 + 1)(dx) \tag{13g}$$
$$\frac{du}{(u^2 + 1)} = (dx) \tag{13h}$$
$$\int \frac{1}{(u^2 + 1)}\,du = \int dx \tag{13i}$$
$$\arctan(u) + c_2 = x + c_1 \tag{13j}$$
$$\arctan(u) = x + (c_1 - c_2) \tag{13k}$$
$$\arctan(u) = x + c \tag{13l}$$
$$\tan(\arctan(u)) = \tan(x + c) \tag{13m}$$
$$u = \tan(x + c) \tag{13n}$$
$$x + y + 1 = \tan(x + c) \tag{13o}$$

The solution is

$y = \tan(x + c) - x - 1$

By analyzing (13a), it is clear that algebraic manipulation will not help transform the problem into a separable form. I begin to apply substitution in (13b) and use the process detailed in (12a) - (12c). The end result is shown in (13h) where I now have a problem that is in separable form. (13i) is the integration step and (13o) shows the reverse substitution step.

Exact Equations

Given an equation in the form:

$$\mathcal{M}(x, y)\, dx + \mathcal{N}(x, y)\, dy = 0 \tag{14}$$

The following technique can be used to solve it. The **first step** requires a test for exactness. If (15) is true, the test has been passed.

$$\frac{\partial \mathcal{M}}{\partial y} = \frac{\partial \mathcal{N}}{\partial x} \tag{15}$$

If the test fails, the **second step** is to try to find an integrating factor to help. An integrating factor is used to convert an inexact differential equation into an exact one. Try $\frac{\mathcal{M}_y - \mathcal{N}_x}{\mathcal{N}}$. If the result is a function of $\mathcal{X}$, then (16) should be used to obtain the integrating factor. If that does not work then try $\frac{\mathcal{N}_x - \mathcal{M}_y}{\mathcal{M}}$ and if the result is a function of $\mathcal{Y}$ then (17) should be used to find obtain the integrating factor.

$$u(x) = e^{\int \frac{\mathcal{M}_y - \mathcal{N}_x}{\mathcal{N}}\, dx} \tag{16}$$

$$u(y) = e^{\int \frac{\mathcal{N}_x - \mathcal{M}_y}{\mathcal{M}}\, dy} \tag{17}$$

The **third step** is performed if an integrating factor had to be used, which is simply multiply both sides of (15) with the integrating factor and then derive (18) and (19)

What (15) indicates is that there is a function f where (18) and (19) are true.

$$\frac{\partial f}{\partial x} = \mathcal{M}(x, y) \tag{18}$$

$$\frac{\partial f}{\partial y} = \mathcal{N}(x, y) \tag{19}$$

There are two approaches for the **fourth step.** In this version of the paper, I present one based on (20)

$$f(x, y) = \int \mathcal{M}(x, y)\, dx + g(y) \tag{20}$$

Differentiate (21) with respect to y and assume the result equals $\mathcal{N}(x, y)$ as performed in (21), (22), and (23)

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int (\mathcal{M}(x, y)\, dx + g(y)) \tag{21}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int (\mathcal{M}(x, y)\, dx) + g'(y) \tag{22}$$

$$\frac{\partial}{\partial y} \int (\mathcal{M}(x, y)\, dx) + g'(y) = \mathcal{N}(x, y) \tag{23}$$

$$g'(y) = \mathcal{N}(x, y) - \frac{\partial}{\partial y} \int \mathcal{M}(x, y)\, dx \tag{24}$$

(24) is obtained from algebraic manipulation (compare (23) and (24)).

The **fifth step** is to find $g(y)$ which requires integrating (24) with respect to y. Plug $g(y)$ into (20) and the solution has been found.

Example for Exact Equation

$$\left(2\, y^2 + 3\, x\right) dx + 2xydy = 0 \tag{25a}$$

$$\frac{\vartheta \mathcal{M}}{\vartheta y} = 4\, y \tag{25b}$$

$$\frac{\vartheta \mathcal{N}}{\vartheta x} = 2\, y \tag{25c}$$

At this point, it is clear that the test for exactness failed. Look for an integrating factor by first evaluating $\frac{\mathcal{M}_y - \mathcal{N}_x}{\mathcal{N}}$. (25h) contains a function only of x.

$$\frac{4y - 2y}{2xy} = \frac{2y}{2xy} \tag{25d}$$

$$\frac{4y - 2y}{2xy} = \frac{1}{x} \tag{25e}$$

$$u(x) = e^{\int \frac{\mathcal{M}_y - \mathcal{N}_x}{\mathcal{N}} dx} \tag{25f}$$

$$u(x) = e^{\int \frac{1}{x} dx} \tag{25g}$$

$$u(x) = x \tag{25h}$$

(25a) is then multiplied by the integrating factor and the result is (25i)

$$\left(2xy^2 + 3x^2\right)dx + 2x^2 y dy = 0 \tag{25i}$$

$$\frac{\partial \mathcal{M}}{\partial y} = 4xy \tag{25j}$$

$$\frac{\partial \mathcal{N}}{\partial x} = 4xy \tag{25k}$$

The exactness test has now been satisfied.

$$f(x, y) = \int \left(2xy^2 + 3x^2\right) dx \tag{25l}$$

$$f(x, y) = x^2 y^2 + x^3 + g(y) \tag{25m}$$

The remaining steps show the approach needed to determine $g(y)$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^2 y^2 + x^3 + g(y)\right] \tag{25n}$$

$$\frac{\partial f}{\partial y} = x^2(2y) + g'(y) \tag{25o}$$

$$\mathcal{N} = x^2(2y) + g'(y) \tag{25p}$$

$$2x^2 y = x^2(2y) + g'(y) \tag{25q}$$

$$g'(y) = 0 \tag{25r}$$

$$g(y) = \int g'(y) dy \tag{25s}$$

$$g(y) = \int 0 \, dy \tag{25t}$$

$$g(y) = 0 \tag{25u}$$

(26) contains the implicit solution that was found.

$$x^2 y^2 + x^3 = c \tag{26}$$

Additional Comments about Exact Equations

Second Approach

The fourth step was implemented using (20) with the assumption that $\frac{\partial f}{\partial x} = M(x, y)$. (27) provides a second approach that can be used to solve exact equations.

$$\frac{\partial f}{\partial y} = \mathcal{N}(x, y) \tag{27}$$

$$f(x, y) = \int \mathcal{N}(x, y)\, d y + b(x) \tag{28}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\int \mathcal{N}(x, y)\, d y + b(x)\right) \tag{29}$$

$$\frac{\partial}{\partial x} \int \mathcal{N}(x, y)\, d y + b'(x) = \mathcal{M}(x, y) \tag{30}$$

$$b'(x) = \mathcal{M}(x, y) - \frac{\partial}{\partial x} \int \mathcal{N}(x, y)\, d y \tag{31}$$

To find $b(x)$, simply integrate $b'(x)$ with respect to x and plug into (28).

Example for Exact Equation (using second approach)

(25i) and (32) are the same and an appropriate starting point

$$\left(2 x y^2 + 3 x^2\right) d x + 2 x^2 y d y = 0 \tag{32}$$

$$f(x, y) = \int 2 x^2 y\, d y + b(x) \tag{33a}$$

$$f(x, y) = x^2 y^2 + b(x) \tag{33b}$$

$$\frac{\partial}{\partial x}\left(x^2 y^2 + b(x)\right) = \left(2 x y^2 + 3 x^2\right) \tag{34}$$

$$b'(x) = \left(2 x y^2 + 3 x^2\right) - 2 x y^2 \tag{35}$$

$$b(x) = \int b'(x)\, d x \tag{36a}$$

$$b(x) = \frac{3 x^3}{3} \tag{36b}$$

$$f(x, y) = x^2 y^2 + x^3 \tag{37}$$

The implicit solution that was found. $\boldsymbol{x^2 y^2 + x^3 = c}$