Solutions of Homogenous Linear Differential Equations with Constant Coefficients.

Introduction

This paper focuses on how to solve homogenous linear differential equations that meet the following criteria:

$$a_n y^n + a_{n-1} y^{n-1} + \ldots + a_1 y' + a_0 y = 0 \tag{1}$$

where $a_{0,} a_{1,} \ldots, a_n$ are constants and $a_n \neq 0$.

Theory

Linear Independence

Linear dependence of functions is defined as

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0 \tag{2}$$

If there exists a set of constants $c_1, c_2,$
$c_3, \ldots c_n$ (not all equal zero) where (1) is true. If no set of constants exists then (1) contains linearly independent functions.

Characteristic Equation

The assumption made is that a solution for (1) will have the form of (3).

$$y = e^{mx} \tag{3}$$

Substitute this assumed value of y into (1) and the following is obtained:

$$a_n \frac{d^n}{dx^n} e^{mx} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} e^{mx} + \ldots + \frac{d}{dx} a_1 e^{mx} + a_{0\, e^{mx}} = 0 \tag{4}$$

Expand the differentiation expressed in (4) and the result is

$$a_n m^n e^{mx} + a_{n-1} m^{n-1} e^{mx} + \ldots + a_1 m e^{mx} + a_{0\, e^{mx}} = 0 \tag{5}$$

Now factor e^{mx} out of (5) and the result is (6)

$$e^{mx} \left(a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0\right) = 0 \tag{6}$$

e^{mx} will never equal 0 so $\left(a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0\right)$ must.

$$\left(a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0\right) = 0 \tag{7}$$

(7) is called the **characteristic equation.** To better understand, lets consider the following second order differential equation.

$$y'' + 6 y' + 5 y = 0 \tag{8a}$$

$$y = e^{mx} \tag{8b}$$

$$y' = m e^{mx} \tag{8c}$$

$$y'' = m^2 e^{mx} \tag{8d}$$

$$m^2 e^{mx} + 6me^{mx} + 5 e^{mx} = 0 \tag{8e}$$

$$e^{mx}(m^2 + 6m + 5) = 0 \tag{8f}$$

$$m^2 + 6m + 5 = 0 \tag{8g}$$

(8b) represents the assumption made about the form the solution will take. (8c) and (8d) are the 1st and 2nd derivatives of (8b). Substituting (8b), (8c), and (8d) into (8a) and the result is (8e). The characteristic equation is shown in (8g).

Approach

The **first step** is to translate the differential equation into the correct characteristic form. The **second step** is to determine the roots of the characteristic equation. The **third step** depends on the type of roots obtained (see each case for specific information). There are three possible outcomes for the roots of the characteristic equation :

(a) All roots are distinct and real.

(b) All roots are real but repeat.

(c) Some or all roots are imaginary.

Case 1 All roots are distinct and real

(8a) and (10a) fall into this category. Both are second order differential equations which as expected will have two roots. The roots are different and consequently will yield two linearly independent solutions which when combined will provide a general solution that looks like (9). c_1 and c_2 are constants that can be found if initial condition information is provided. m_1 and m_2 represent the roots of the characteristic equation.

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} \tag{9}$$

Example of all roots are distinct and real

$$y'' - 3y' + 2 = 0 \tag{10a}$$

$$m^2 - 3m + 2 = 0 \tag{10b}$$

$$(m-2)(m-1) = 0 \tag{10c}$$

$$y_c = c_1 e^{x} + c_2 e^{2x} \tag{10d}$$

(10b) shows the translation from differential equation to characteristic equation **(first step)**. The power of m (in 10b) matches the derivative number of y (in 10a). and notice the preservation of the coefficients. The same result would have been obtained using the process shown in steps (8b) through (8g). (10c) (**second step**) shows the factors for (10b) obtained by using the quadratic equation. The **last step** simply requires substituting the roots found into (9) to obtain the general solution (10d).

Case 2 All roots are real but repeat.

With repeating roots, the result is two linearly dependent solutions which when combined look like (11) and this is unacceptable. The general solution has to contain linearly independent components, to achieve this use (12). (12) provides a way to handle differential equations of order two or more. This is also used for Case 3 when there are repeating roots and imaginary roots.

$$y_c = c_1 e^{mx} + c_2 e^{mx} \tag{11}$$

$$y_c = \left(c_1 + c_2 x + c_3 x^2 + \ldots + c_n x^{n-1}\right) e^{ax} \tag{12}$$

To see how to use (12) lets assume that the following roots have been obtained: $(m)(m-a)^2 (m-b)^3$. This translates to $m = 0$ (once), $m = a$ (twice), and $m = b$ (three times). Using (12) as a guide, yields (13).

$$y_c = c_1 + (c_2 + c_3\, x)\, e^{a x} + \left(c_4 + c_5\, x + c_6\, x^2\right) e^{\mathrm{b} x} \tag{13}$$

Note: The terms that contain x (in our example, the case where m = a and m =b), the highest exponent of x is equal the number of times the root is repeated (n) minus 1. The trick it to apply (12) for each root. Also note that c_1 does not have an e term because e^0 is one.

Example all roots are real but repeat.

$$y'' + 4\,y' + 4\,y = 0, \quad y(0) = 1, \; y'(0) = 1 \tag{14a}$$

$$m^2 + 4\,m + 4 = 0 \tag{14b}$$

$$(m+2)\,(m+2) = 0 \tag{14c}$$

$$y_c = c_1\, e^{-2\,x} + c_2\, x\, e^{-2\,x} \tag{14d}$$

$$y_c' = \left(-2\,c_1\, e^{-2\,x} + c_2 - 2\,c_2\, x\right) e^{-2\,x} \tag{14e}$$

$$c_1 = 1 \tag{14f}$$

$$-2\,c_1 + c_2 = 1 \tag{14g}$$

$$c_2 = 3 \tag{14h}$$

$$y_p = (1 + 3\,x)\, e^{-2\,x} \tag{14i}$$

This example differs from the previous one because it contains repeating roots and it has initial conditions.

I follow the same process of equation translation, and root finding as done for case 1 but in (14d) I use (12). The root value m = -2 appears twice (n=2). The derivative the general solution is needed (14e) so that initial value conditions can be used to calculate the particular solution.

Case 3 Some or All roots are imaginary

The roots in this case look like $\alpha \pm \beta$i (you will always get conjugate pairs). Technically this is like case 1 because the roots are distinct. The difference is how the general solution is written (see 15).

Example some or all roots are imaginary

$$y_c = c_1\, e^{\alpha x} \cos(\beta x) + c_1\, e^{\alpha x} \sin\beta x) \tag{15}$$

$$y'' - 2\,y' + 5\,y = 0 \tag{16a}$$

$$m^2 - 2\,m + 5 = 0 \tag{16b}$$

$$m = 1 + 2\,i \quad \text{or}\; m = 1 - 2\,i \tag{16c}$$

$$y_c = e^x (\,c_1 \cos(2\,x) + c_2 \sin(2\,x)) \tag{16d}$$