Classification Scheme for the Phase Portraits of Linear Systems. (Draft)

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Introduction

Phase portraits are used to provide qualitative information about linear and non-linear differential equations. The goal of this paper is to explain the scheme used to classify phase portraits of two dimensional linear systems. Phase portraits allow us to gain insight about a differential system even when no other method is available. A trajectory is the primary component of a phase portrait. A trajectory can be thought of as the path taken from an initial point to an end point. The initial point is represented by the initial condition of the differential equation. When a differential equation is provided without an initial condition, the portrait will show infinitely many trajectories. The phase portraits for the Romeo and Juliet love affair show the initial condition point and a subset of the surrounding phase plane.

The shapes of the portraits shown in this paper represent general categories. It is possible to see linear phase portraits that meet the criteria described below but vary in shape (somewhat distorted) or orientation when viewed on the phase plane. It has been said that a picture is worth a 1000 words. Using that concept as a guideline, I have tried to keep the mathematics to an absolute minimum.

Math foundation

$$\dot{\mathcal{R}} = a\mathcal{R} + b\mathcal{J} \tag{1}$$

$$\dot{\mathcal{J}} = c\mathcal{R} + d\mathcal{J} \tag{2}$$

$$\begin{pmatrix} \dot{\mathcal{R}} \\ \dot{\mathcal{J}} \end{pmatrix} = \begin{pmatrix} \mathcal{R} \\ \mathcal{J} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3}$$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ab) - (c\,d) \tag{4}$$

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a - \lambda)(\mathrm{d} - \lambda) - (bc) \tag{5a}$$

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \lambda^2 - \lambda(a+d) + (ad) - (bc) \tag{5b}$$

$$\lambda^2 - \lambda a - \lambda d + (ad) - (bc) = 0 \tag{5c}$$

$$\lambda_1 = \frac{(a + \mathrm{d}) - \sqrt{a^2 - 2\,ad + d^2 + 4\,bc}}{2} \quad (5d)$$

$$\lambda_2 = \frac{1}{2}\left((a+d) + \sqrt{a^2 - 2\,ad + d^2 + 4\,bc}\right) \quad (5e)$$

$$\mathcal{X}(t) = c_1 \times k_1 \times e^{\lambda_1 t} + c_2 \times k_2 \times e^{\lambda_2 t} \quad (6)$$

$$\Delta = \lambda_1 * \lambda_2 \quad (7)$$

$$\tau = \lambda_1 + \lambda_2 \quad (8)$$

(3) is a matrix representation for (1) and (2). (4) defines the determinant based on the parameters (coefficients) of (1) and (2). In order to solve the problem, we need to obtain the characteristic equation. The process begins with (5) where the variable λ is introduced. The solution to (5c) involves using the quadratic equation and after some algebraic manipulation we obtain (5d) and (5e). The variables λ_1 and λ_2 are called eigenvalues.

(6) is the general solution. In order to find the specific solution, k_1 and k_2 must first be obtained. k_1 and k_2 are called eigenvectors. This happens through the use of Gaussian elimination. Once obtained the next step is to solve for c_1 and c_2 which requires using the initial conditions, eigenvectors and solving a simple algebraic system.

(7) and (8) represent the determinant(Δ) and trace (τ) of the matrix $\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$. **It is on the τ Δ plane that the classification of two dimensional phase portraits begins**. It is important to note that the determinant calculations used in this paper are for a 2 x 2 matrix. (4) shows the deteminant (for $n = 2$) as defined in many linear algebra textbooks. The determinant for a $n \times n$ matrix (where $n >= 2$) can be found using the Laplace expansion.

Solved Problem

$$\dot{r} = 5\,r + 10\,\mathrm{j} \quad (9)$$

$$\dot{j} = 10\,r + 5\,\mathrm{j} \quad (10)$$

$$\begin{pmatrix} \dot{r} \\ \dot{j} \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 10 & 5 \end{pmatrix} \begin{pmatrix} r \\ j \end{pmatrix} \quad (11)$$

$$\det\begin{pmatrix} 5-\lambda & 10 \\ 10 & 5-\lambda \end{pmatrix} = \det\begin{pmatrix} 5 & 10 \\ 10 & 5 \end{pmatrix} - \lambda \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (12)$$

$$\det\begin{pmatrix} 5-\lambda & 10 \\ 10 & 5-\lambda \end{pmatrix} = (5-\lambda)(5-\lambda) - (10)(10) \quad (13)$$

$$\det\begin{pmatrix} 5-\lambda & 10 \\ 10 & 5-\lambda \end{pmatrix} = \lambda^2 - 10\,\lambda - 75 \quad (14)$$

$$\lambda_1 = 15 \quad (15)$$

$$\lambda_2 = -5 \quad (16)$$

$$\Delta = -75 \quad (17)$$

$$\tau = 10 \quad (18)$$

Case λ_1

$$\begin{pmatrix} 5-\lambda & 10 \\ 10 & 5-\lambda \end{pmatrix} = \begin{pmatrix} 5-15 & 10 \\ 10 & 5-15 \end{pmatrix} \quad (19)$$

$$\begin{pmatrix} 5-15 & 10 \\ 10 & 5-15 \end{pmatrix} = \begin{pmatrix} -10 & 10 \\ 10 & -10 \end{pmatrix} \quad (20)$$

$$-10\,k_1 + 10\,k_2 = 0 \quad (21)$$

$$10\,k_1 - 10\,k_2 = 0 \quad (22)$$

Through the use of Gaussian Elimination, it is determined that $k_1 = 1$ and that $k_2 = 1$

Case λ_2

$$\begin{pmatrix} 5-(-5) & 10 \\ 10 & 5-(-5) \end{pmatrix} = \begin{pmatrix} 10 & 10 \\ 10 & 10 \end{pmatrix} \tag{23}$$

$$10\,k_1 + 10\,k_2 = 0 \tag{24}$$

$$10\,k_1 + 10\,k_2 = 0 \tag{25}$$

Through the use of Gaussian Elimination, it is determined that $k_1 = 1$ and that $k_2 = -1$

At this point, we have determined the following about the solution: $\mathcal{X}(\mathrm{t}) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{15\,t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-5\,t}$

Values for c_1 and c_2 must be found. This example has no initial conditions so the solution given is sufficient. If I had initial conditions, the problem would be structured as done on (26) where the initial conditions are placed in matrix form along with the eigenvectors. The next step would be to convert from matrix form to algebraic form as done on (27) and (28). The last step involves some straight forward algebra. Figure 1 is the phase portrait **(a Saddle Point)** for this example.

$$\begin{pmatrix} r_0 \\ j_0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{26}$$

$$r_0 = c_1 - c_2 \tag{27}$$

$$j_0 = c_1 + c_2 \tag{28}$$

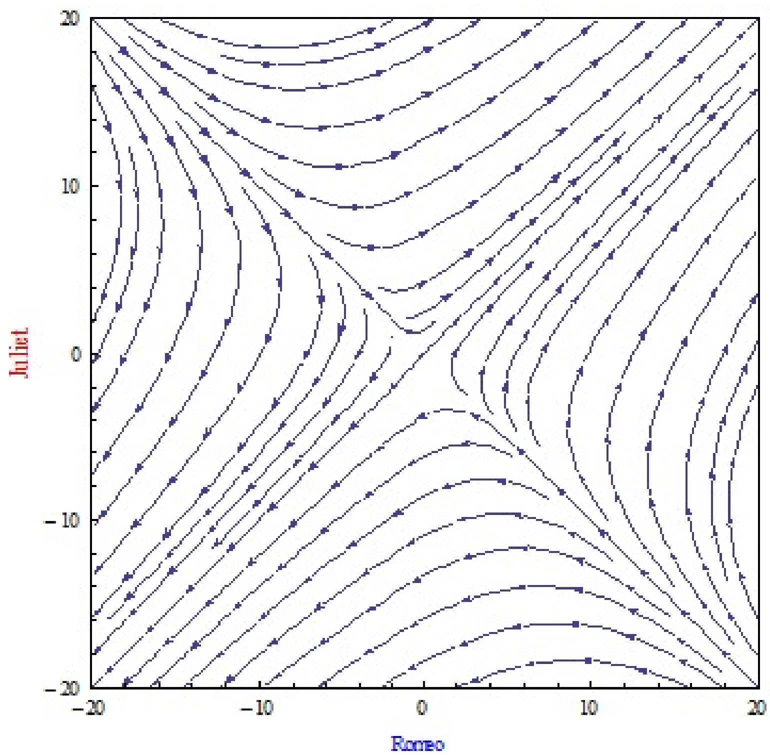

Figure 1.

Critical Points

A phase portrait's form is determined largely by the type of critical point that is encountered. A critical point is defined as the location where $\frac{j'}{r'} = \frac{0}{0}$. Without having to perform the calculations needed to obtain a specific solution, it is possible to learn much about a linear system by examining the eigenvalues, trace and determinant. The following provides a straight forward guide to determining the type of critical point that exists

Case 1 Real Distinct Eigenvalues $(\lambda_1 \neq \lambda_2)$

Stable Node

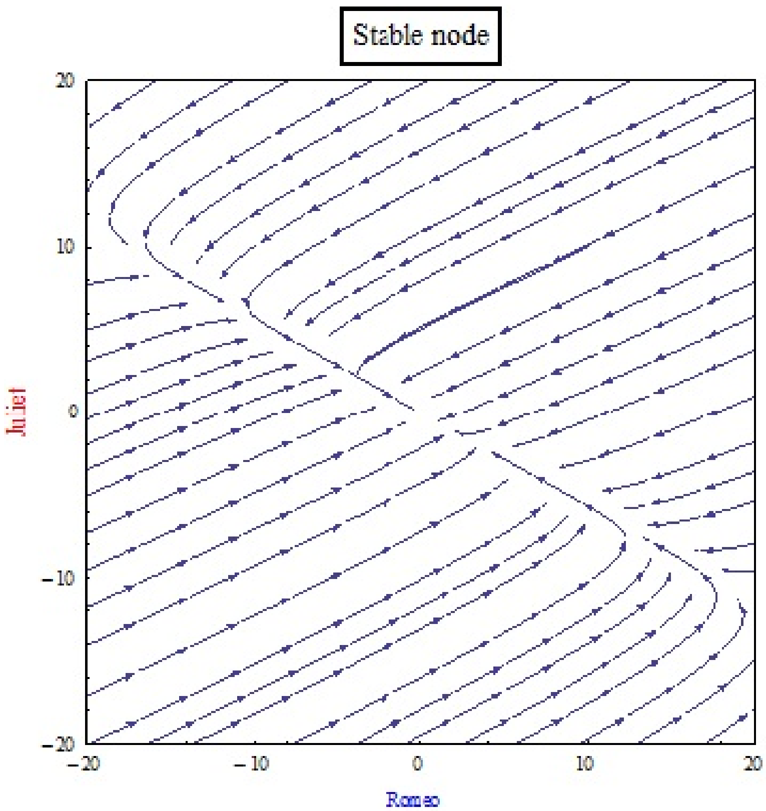

Figure 2.

A stable node exists when both eigenvalues are negative. This mean the following conditions have been met: $\tau^2 - 4\Delta > 0,\ \tau < 0,\ \Delta > 0$.

Unstable Node

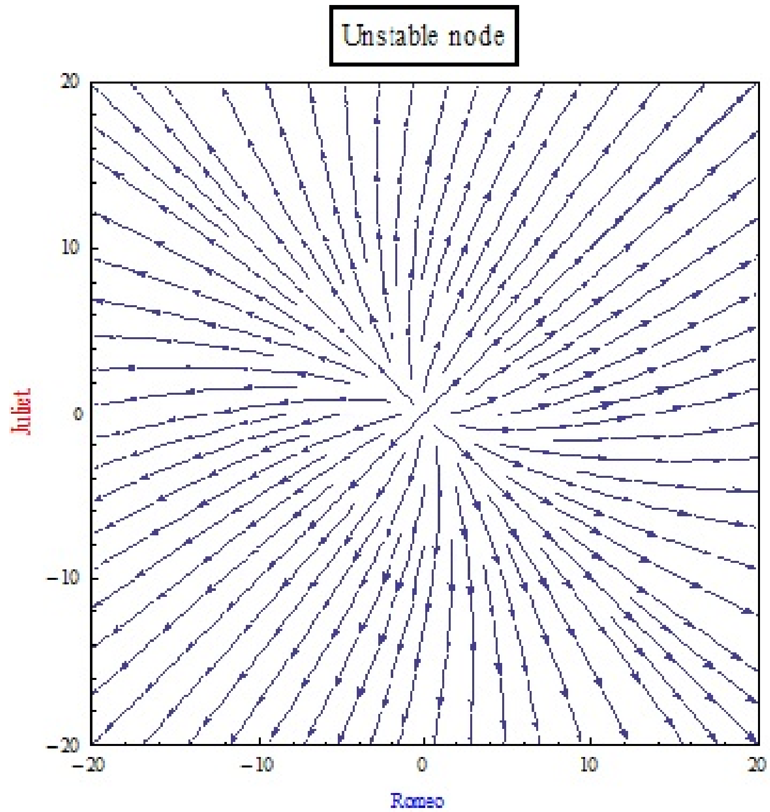

Figure 3.

An unstable node occurs when both eigenvalues are positive. This means the following conditions have been met: $\tau^2 - 4\Delta > 0$, $\tau > 0$, $\Delta > 0$.

Saddle Point

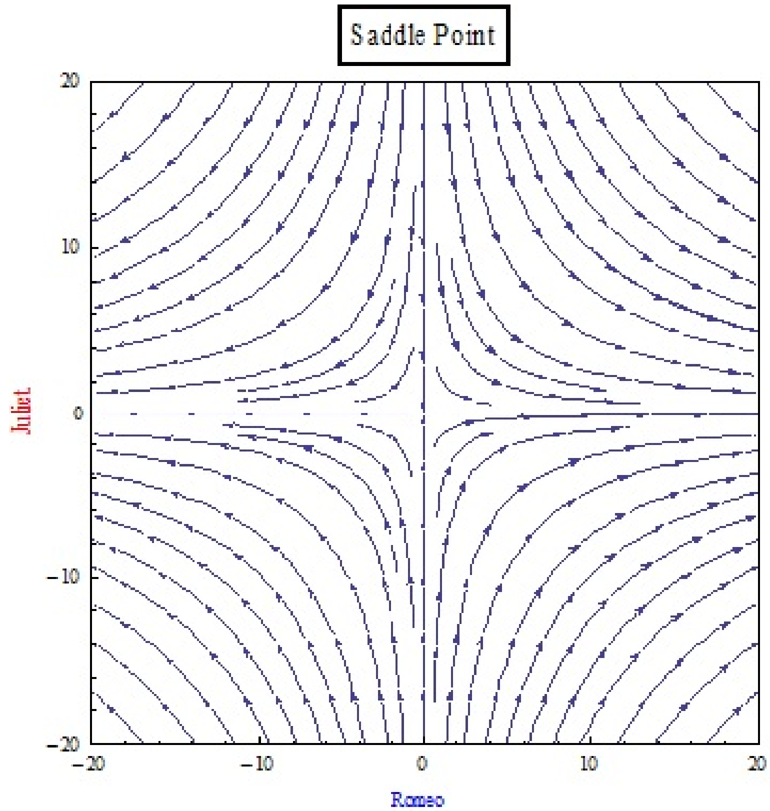

Figure 4.

Saddle points occur when the real eigenvalues have opposite signs.

Exception

The key to the λ values in this section is that they are distinct real values. It is possible to encounter a scenario where: $\tau^2 - 4\Delta > 0$, and $\Delta = 0$. This can occur when one of the λ values equals 0 and the other does not. The fact that $\tau^2 - 4\Delta > 0$ is true does not classify the resulting phase portrait as a node. It turns out that in Figures 5 and 6 , there is not a single critical point but rather an entire line of critical points as indicated by the thick black line.

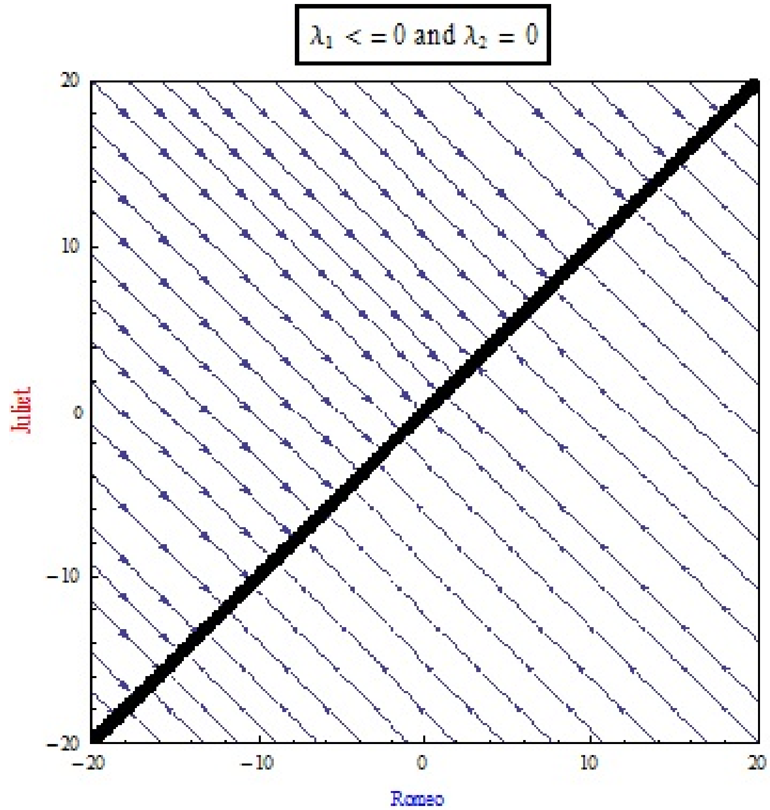

Figure 5.

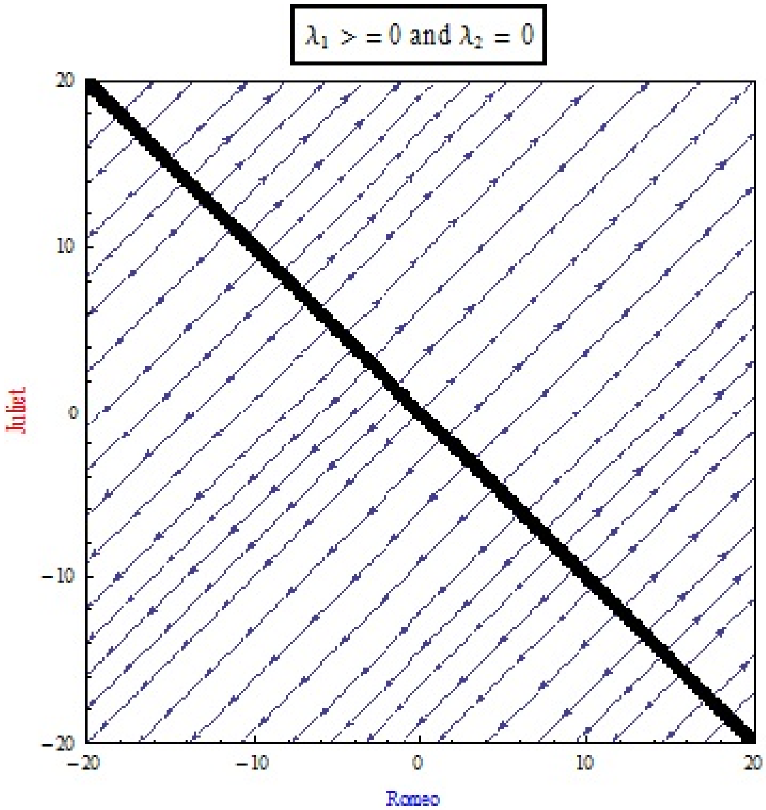

Figure 6.

Case 2 Repeated Real Eigenvalues $(\lambda_1 = \lambda_2)$

The parabola $\tau^2 - 4\Delta = 0$ contains degenerate nodes and star nodes.

Degenerate Stable node

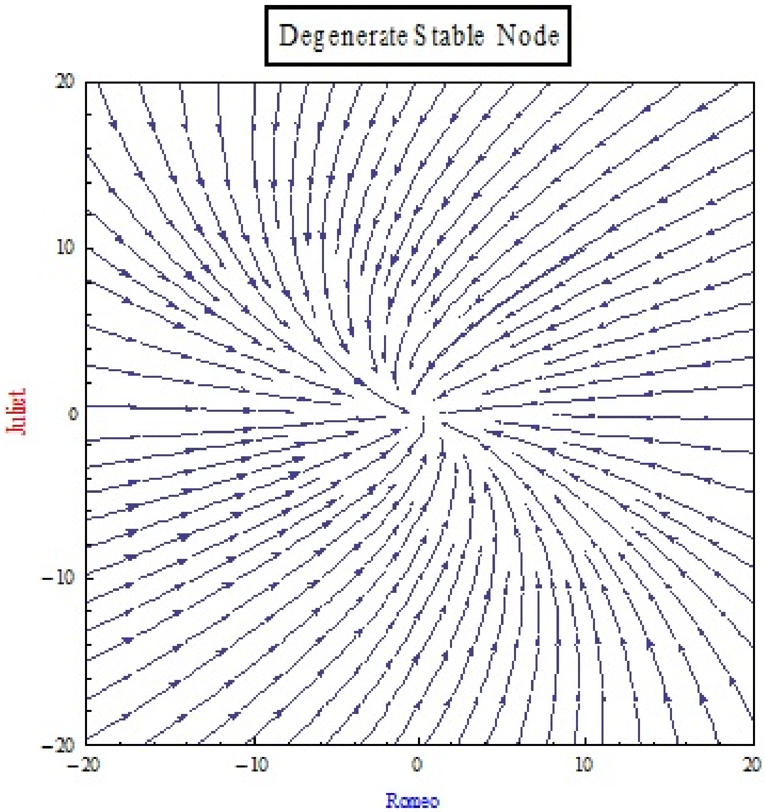

Figure 7.

When $\lambda_1 < 0$, a degenerate stable node occurs. The trajectories all head toward the critical point.

Degenerate Unstable node

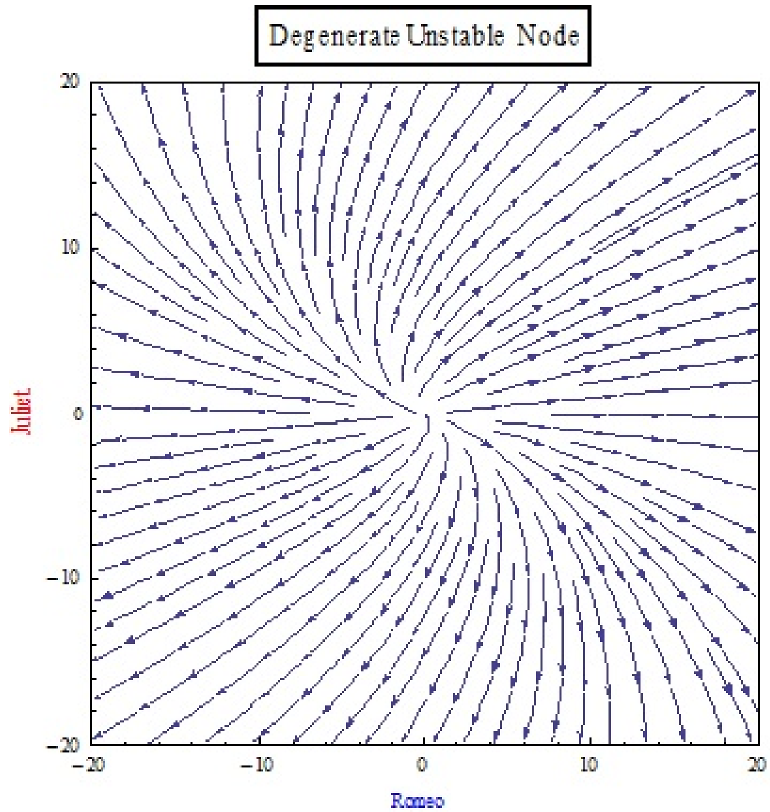

Figure 8.

When $\lambda_1 > 0$, a degenerate stable node occurs. All of the trajectories are heading away from the critical point.

Star Node

A star node will have repeated real eigenvalues but the differential system in matrix form looks like:

$$\begin{pmatrix} \dot{r} \\ \dot{j} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} r \\ j \end{pmatrix}$$

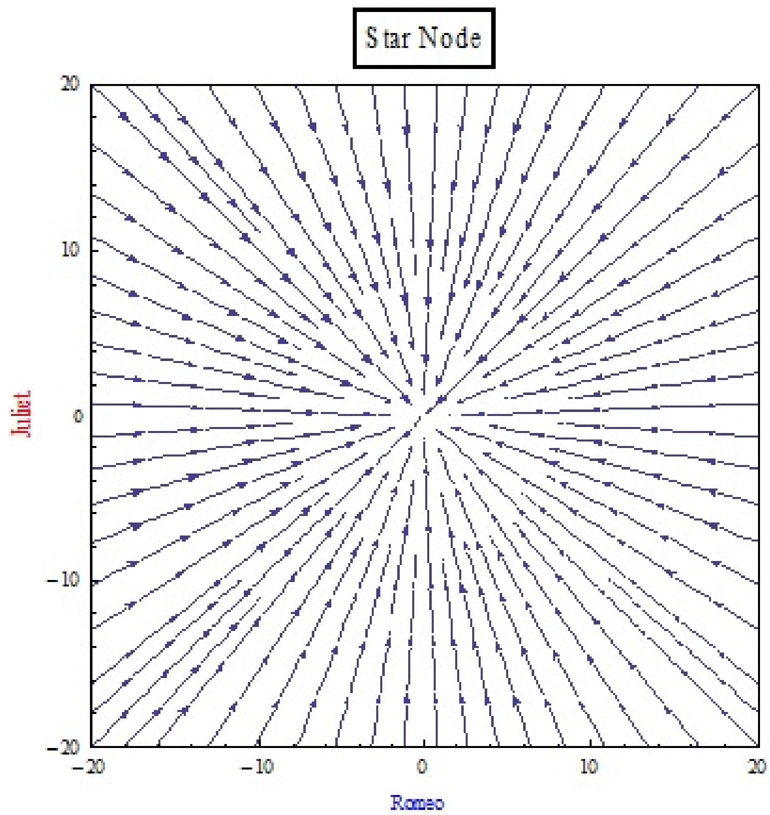

Figure 9.

Case 3 Complex Eigenvalues ($\tau^2 - 4\Delta < 0,\ \lambda_1 = \alpha$ - iβ, $\lambda_2 = \alpha$ + iβ)

Center

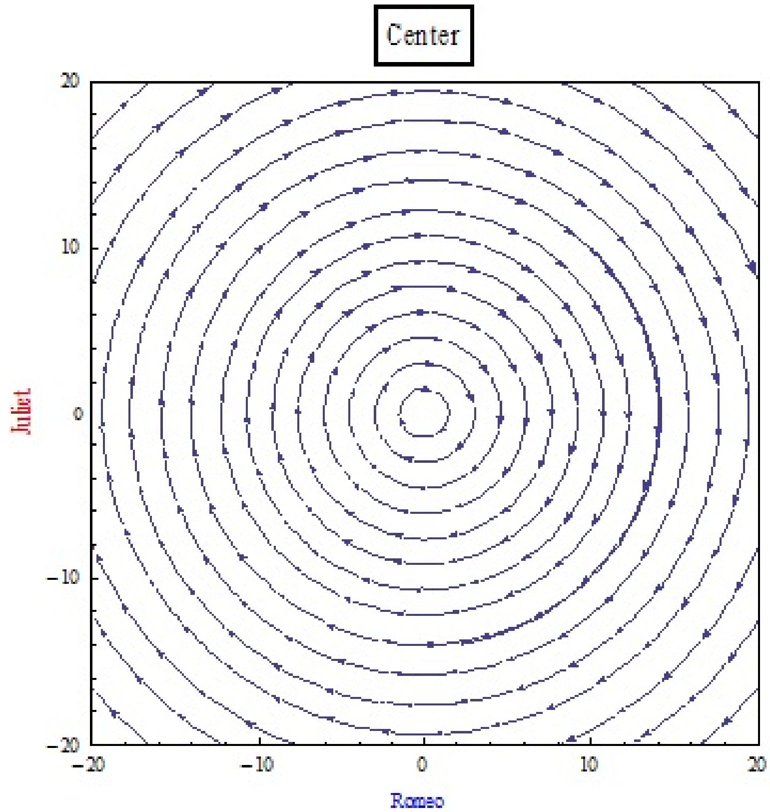

Figure 10.

A center occurs when both λ_1 and λ_2 only have imaginary components ($\alpha = 0$).

Stable Spiral

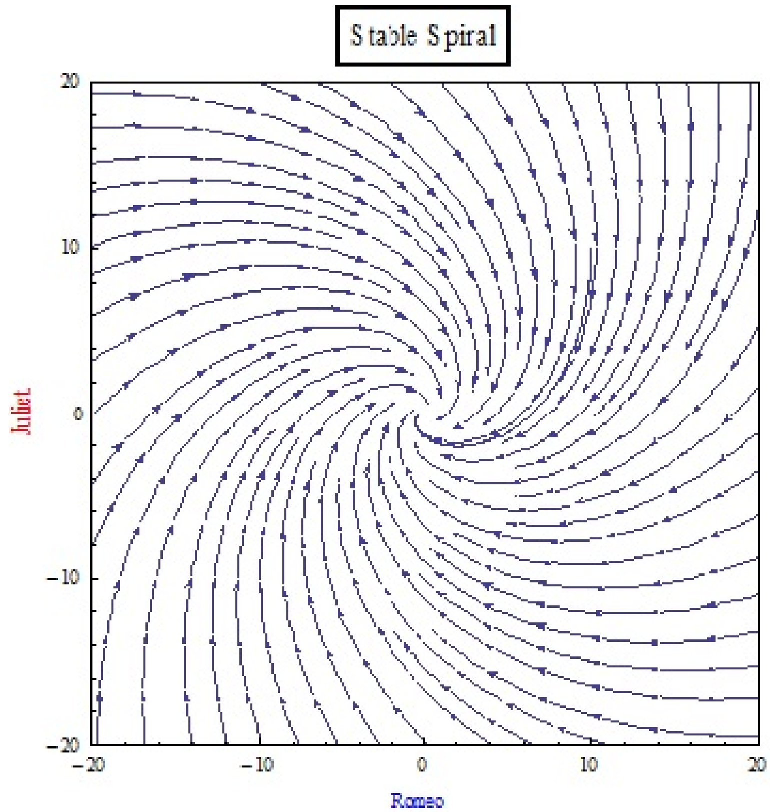

Figure 11.

When the real component in λ_1 and λ_2 is less than 0, a stable spiral occurs. It is stable because all of the trajectories flow toward the critical point.

Unstable Spiral

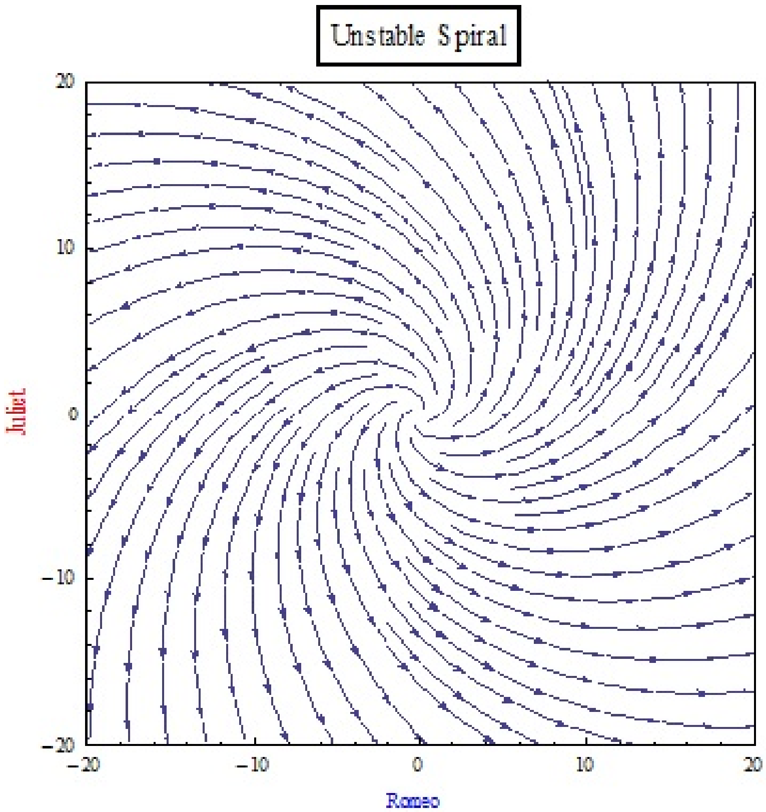

Figure 12.

When the real component in λ_1 and λ_2 is greater than 0, an unstable spiral occurs. It is unstable because all trajectories flow away from the critical point.

Phase Portrait By Hand

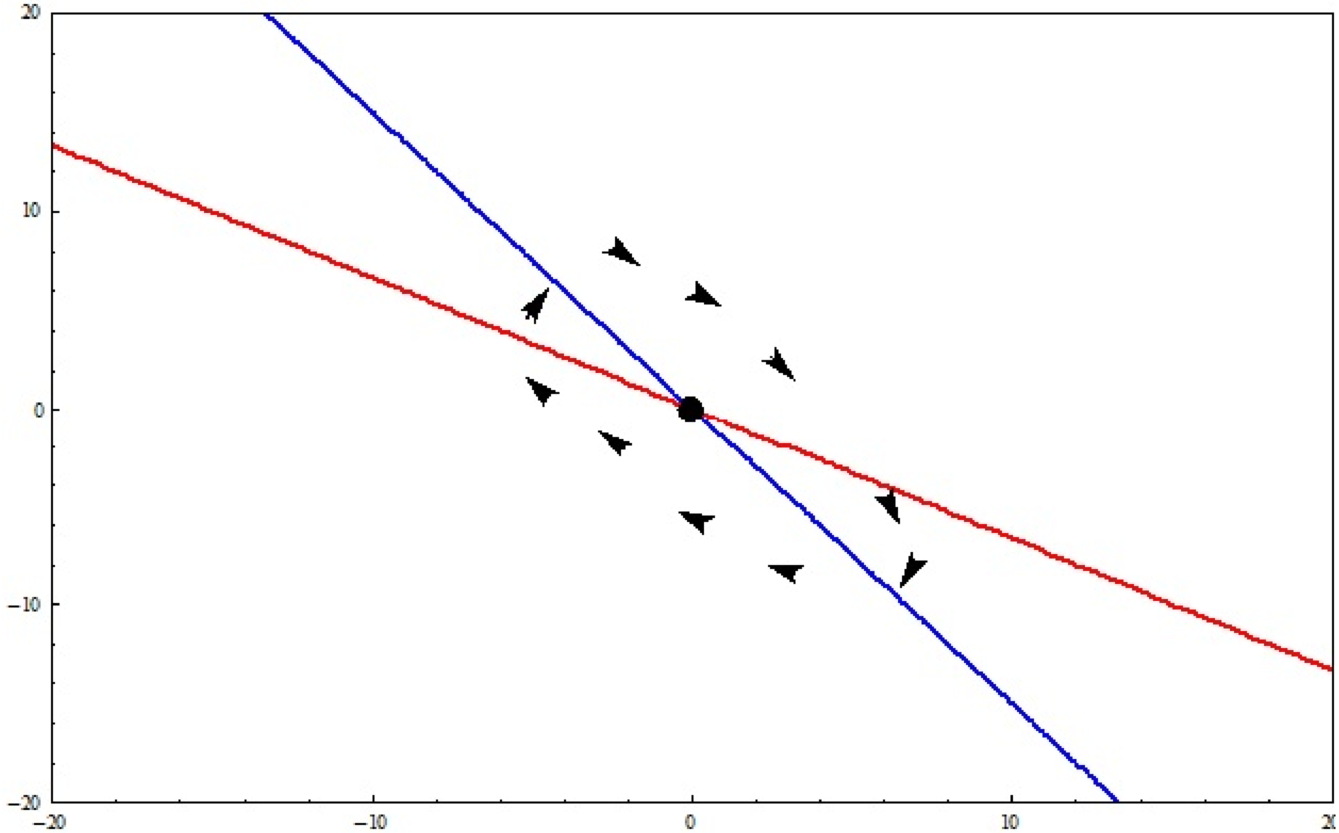

Figure 13.

r	j	dr/dt	dj/dt
1	5	17	-13
3.815	2.659	15.607	-16.763
7.514	-3.507	4.507	-15.528
7.36	-6.667	-5.281	-8.746
4.046	7.514	14.45	2.89
1.811	-6.667	-16.379	7.901
-0.1927	-5.511	-16.9184	11.6001
-1.349	-4.74	-16.918	13.527
-7.129	6.898	6.436	7.591
0.03468	5.511	16.6024	-11.126

Figure 14.

Figure 13 is a freehand attempt to draw a phase portrait.The approach used can also be done with non-linear differential equations (there are a few more steps that I have not mentioned). The equations shown in (29) and (30) are based on the model example (Fire and water) posted on the website. It is important to understand that I did not concern myself with an initial condition. My goal was simply to get an approximate understanding of what was happening on the phase plane. I choose various values of r and j from the first, second, third, and fourth quadrants and plugged them directly into equations (29) and (30). Figure 14 contains the values obtained. At each (r,j) point, I looked at the slope value and drew an arrow in that direction.

$$\dot{\mathcal{R}} = 2\,\mathcal{R} + 3\,\mathcal{J} \quad (29)$$

$$\dot{\mathcal{J}} = -3\,\mathcal{R} - 2\,\mathcal{J} \quad (30)$$

Normal Form

The following has been included to show how to reduce a linear ordinary differential equation (ode) to a lower order. In this example, I start with a 2nd order ode and reduce it to a 1st order ode. This is done by introducing the variable y, differentiating it, and performing substitution. This makes it possible to transform a differential equation into a simpler form for qualitative analysis.

$$m\,\frac{d^2 x}{dt^2} + r\,\frac{dx}{dt} + kx = 0 \quad (31)$$

$$\frac{dx}{dt} = y \quad (32)$$

$$\frac{dy}{dt} = \frac{d}{dt}\left[\frac{dx}{dt}\right] = \frac{d^2 x}{dt^2} \quad (33)$$

(34)

$$m\,\frac{dy}{dt} + ry + kx = 0 \quad (35)$$

References

Belykh, Igor. “Nonlinear Dynamics and Chaos”. Georgia State University. 2013. Lecture.

Stogratz, Stevens. (2015). “Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering”. Westview Press.