Power Series (Draft)

Introduction

This paper demonstrates how to use approximation techniques to solve linear differential equations of the following type:

$$y^n + f_{n-1}(x)\, y^{(n-1)} + \dots + f_1(x)\, y' + f_0(x)\, y \;=\; Q(x) \tag{1}$$

The two approaches described require initial conditions so that a particular solution can be found. Both approaches can be extended to solve a system of first order linear differential equations like the Romeo and Juliet problems.

Theory and Definition

Definition 1 Taylor series

$$f(x) \;= f(x_0) \;+\; f'(x_0)\,(x - x_0) \;+\; \frac{f''(x_0)\,(x - x_0)^{\,2}}{2!} \;+\; \dots \;+\; \frac{f^n(x_0)\,(x - x_0)^n}{n!} \;+\; \dots \tag{2}$$

Definition 2 Maclaurin series

$$f(x) \;= f(0) \;+\; f'(0)\,(x) \;+\; \frac{f''(0)\,(x)^{\,2}}{2!} \;+\; \dots \;+\; \frac{f^n(0)\,(x)^n}{n!} \;+\; \dots \tag{3}$$

This difference between (2) and (3) is that $x_0 = 0$. The examples provided will have solutions based on the Maclaurin series because the stated initial conditions have $x = 0$, when x is non-zero then the Taylor series should be used. The Maclaurin series is a special case of the Taylor series.

The following theorems are stated without proof.

Theorem 1

If $f(x)$ and $g(x)$ are defined by a power series when

$$f(x) \;= \sum_{n=0}^{\infty} a_n (x - x_0)^{\,n} \quad \text{and} \tag{4}$$

$$g(x) \;= \sum_{n=0}^{\infty} b_n (x - x_0)^{\,n} \;\text{ then }\; a_0 = b_0,\; a_1 = b_1,\; a_2 = b_2,\; \dots \tag{5}$$

Theorem 2

If $f(x)$ is defined to be a power series then

$$a_0 = f(x_0),\ a_1 = f'(x_0),\ a_2 = \frac{f''(x_0)}{2!}\ \ ...,\ a_n = \frac{f^n(x_0)}{n!} \tag{6}$$

Solution Approaches

Undetermined Coefficients Example

$$y' - xy + x^2 = 0 \quad \text{where} \quad y(0) = 2 \tag{7}$$

Remember that a solution in the power of x has the form (replacing $f(x)$ with $y(x)$ and expanding terms):

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ ... \tag{8}$$

(8) provides a way to define y but y' is needed also, so take the derivative of (8) and now y' is defined. If there was a y'' in the problem then it would be necessary to take the derivative of (9),

$$y' = a_1 + 2 a_2 x^1 + 3 a_3 x^2 + 4 a_4 x^3 + \ ... \tag{9}$$

Substitute (8) and (9) into (7) and obtain:

$$\left[a_1 + 2 a_2 x^1 + 3 a_3 x^2 + 4 a_4 x^3\right] - x\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4\right] + x^2 = 0 \tag{10}$$

Group and simplify to get

$$a_1 + x(2 a_2 - a_0) + x^2(1 + 3 a_3 + a_1) + x^3(4 a_4 + a_2) + x^4(5 a_5 + a_3) = 0 \tag{11}$$

Note: the initial conditions state that when $x = 0$ (Maclaurin), that $y = 2$. Because of Theorem 2 , we know that $a_0 = 2$. Using an identity property, set the coefficients of the powers of x to zero and then solve.

$a_1 = 0$	$2 a_2 - a_0 = 0$	$1 + 3 a_3 + a_1 = 0$	$4 a_4 + a_2 = 0$
	$2 a_2 = a_0$	$1 + 3 a_3 - 0 = 0$	$4 a_4 = a_2$
	$2 a_2 = 2$	$3 a_3 = -1$	$4 a_4 = 1$
	$a_2 = 1$	$a_3 = -\frac{1}{3}$	$a_4 = \frac{1}{4}$

Substitute the a_* values into (8) and the power series approximation of the solution is

$$y = 2 + x^2 - \frac{1}{3} x^3 + \frac{1}{4} x^4 \tag{12}$$

The approximation can be improved by expanding (8) and then performing (9) through (12).

Successive Derivatives Example

Using the same problem as the undetermined coefficient example:

$$y' - xy + x^2 = 0 \quad \text{where} \quad y(0) = 2$$

Change form to

$$y' - xy = -x^2 \tag{13}$$

We see that $Q(x) = -x^2$ and $f_1(x) = -x$. Please note that the initial conditions state that when $x = 0$ that $y = 2$, therefore plug these values into (13) and obtain

$$y' - (0)(2) = 0 \tag{14}$$

Now take the derivative of (13) and obtain

$$y'' - xy' - y = -2x \tag{15}$$

Plug in the initial values and the derived value of y' into (14) and obtain

$$y'' - 0 - 2 = 0$$

$$y'' = 2 \tag{16}$$

Next take the derivative of (15)

$$y''' - xy'' - 2y' = -2. \tag{17}$$

Plug in the initial condition value of y, y' and y'' into (17):

$$y''' - (0)(2) - 2(0) = -2$$

$$y''' = -2$$

Now take the derivative of (17)

$$y^{(4')} - \text{x}y''' - 3y'' = 0 \tag{18}$$

Plug in the initial condition value of y, y', y'' and y''' into (18):

$$y^{(4')} - 0(2) - 3(2) = 0$$

$$y^{(4')} = 6$$

Substitute the values for y, y', y'', y''' and $y^{(4')}$ into the Maclaurin series and obtain

$$y(x) = 2 + \frac{2x^2}{2!} - \frac{-2x^3}{3!} + \frac{6x^4}{4!} = 2 + x^2 - \frac{-x^3}{3} + \frac{x^4}{4}$$