Approximate solution for System of 1st Order Differential Equations

Draft

Introduction

This paper demonstrates how to obtain approximate solutions using a power series technique.

Successive Derivative Approach

Note: For this type of problem, it is important to correctly identify the independent variable. In the example problem, x and y are both functions of t, consequently; differentiation is done with respect to t and not x or y.

Because my initial conditions start at $t = 0$, my goal is to have a series expansion of the form:

$$x(t) = x(0) + x'(0)\,t + \frac{x''(0)\,t^2}{2!} + \frac{x'''(0)\,t^3}{3!} + \ldots \tag{1}$$

$$y(t) = y(0) + y'(0)\,t + \frac{y''(0)\,t^2}{2!} + \frac{y'''(0)\,t^3}{3!} + \ldots \tag{2}$$

The example problem is (3).

$$\frac{dx}{dt} = hx - kxy \tag{3}$$

$$\frac{dy}{dt} = kxy - py$$

Where h, k, and p are constants and

$x(0) = 1,\ y(0) = 1$

The first step is to make sure that I am working with a system of first order differential equations. Once confirmed, the second step is to start successively differentiating $\frac{dx}{dt}$. Starting with x', I find x'' then x'''. I do this because I want four terms of approximation for $x(t)$. If I wanted five terms of approximation, then I would differentiate x''' to obtain $x^{(\mathrm{iv})}$. The third step is to start successively differentiating $\frac{dy}{dt}$ to obtain y'' and y'''. Again, I desire four terms of approximation for $y(t)$.

$$x'' = hx' - \mathrm{k}x'\,y - kxy' \tag{4}$$

$$x''' = hx'' - \mathrm{k}x''\,y - \mathrm{k}x'\,y' - \mathrm{k}x'\,y' - kxy'' \tag{5}$$

$$= hx'' - k(x''\,y + 2\,x'\,y' + xy'')$$

$$y'' = \mathrm{k}x'\,y + \mathrm{k}xy' - py' \quad (6)$$

$$y''' = \mathrm{k}x''\,y + \mathrm{k}x'\,y' + \mathrm{k}x'\,y' + kxy'' - py'' \quad (7)$$

$$= \mathrm{k}x''\,y + 2\,kx'\,y' + kxy'' - py''$$

The fourth step is to find the coefficients of (1) and (2). To do this, I use the initial conditions of (3) and evaluate x', y', x'', y'', x''' , and y''' (note: order of evaluation is important.) For the example problem, (8) and (9) represents the results of my evaluation.

$$x' = h - k \quad (8)$$

$$x'' = h[h-k] - k[h-k] - k[k-p]$$

$$= h^2 - hk - hk + k^2 - k^2 + kp$$

$$= h^2 - 2\,hk + kp$$

$$x''' = h\left[h^2 - 2\,hk + kp\right] - (k)\left[h^2 - 2\,hk + kp + 2\,(h-k)\,(k-p) + k(h-k) + k(k-p) - p(k-p)\right]$$

$$= h^3 - 2\,h^2\,k + hkp - k\left[h^2 + kp - 2\,hp - 2\,k^2 + hk + p^2\right]$$

$$= h^3 - 3\,h^2\,k + 3\,hkp - k^2\,p + 2\,k^3 - hk^2 - kp^2$$

$$y' = k - p \quad (9)$$

$$y'' = k(h-k) + k(k-p) - p(k-p)$$

$$= hk - k^2 + k^2 - kp - kp + p^2$$

$$= hk - 2\,kp + p^2$$

$$y''' = (k)\left[h^2 - 2\,hk + kp\right] + (2)\,(k)\,(h-k)\,(k-p) + k\left[hk - 2\,kp + p^2\right] - p\left[hk - 2\,kp + p^2\right]$$

$$= h^2\,k - 2\,hk^2 + k^2\,p + 2\,hk^2 - 2\,hkp - 2\,k^3 + 2\,k^2\,p + hk^2 - 2\,k^2\,p + kp^2 - hkp + 2\,kp^2 - p^3$$

$$= h^2\,k + \mathrm{h}k^2 + k^2\,p - 3\,hkp - 2\,k^3 + 3\,kp^2 - p^3$$

The final step is to substitute the values found in (8) and (9) into (1) and (2) which yields (10) and (11).

$$x(t) = 1 + (h-k)\,(t) + \frac{\left(h^2 - 2\,hk + kp\right)\times t^2}{2!} + \frac{\left(h^3 - 3\,h^2\,k + 3\,hkp - k^2\,p + 2\,k^3 - hk^2 - kp^2\right)\times t^3}{3!} + \ldots \quad (10)$$

$$y(t) = 1 + (k-p)\,(t) + \frac{\left(hk - 2\,kp + p^2\right)\times t^2}{2!} + \frac{\left(h^2\,k + \mathrm{h}k^2 + k^2\,p - 3\,hkp - 2\,k^3 + 3\,kp^2 - p^3\right)\times t^3}{3!} + \ldots \quad (11)$$