Variation of Parameters.

Technique for solving High order nonhomogeneous linear differential equations.

Introduction

This paper focuses on how to solve nonhomogeneous linear differential equations that meet the following criteria:

$$a_n y^n + a_{n-1} y^{n-1} + \ \dots \ + a_1 y' + a_0 y = Q(x) \quad \text{where } a_{0,} a_{1,} \dots, a_n \text{ are constants, and } a_n \neq 0, \text{ and } Q(x) \text{ can have infinite linearly independent derivatives.} \tag{1}$$

It is also applicable to nonhomogeneous linear differential equations that meet the following criteria:

$$f_n y^n + f_{n-1} y^{n-1} + \ \dots \ + f_1 y' + f_0 y = Q(x) \quad \text{where } f_{0,} f_{1,} \dots, f_n \text{ are functions, and } f_n \neq 0, \text{ and } Q(x) \text{ can have infinite linearly independent derivatives.} \tag{2}$$

Overview

For the sake of clarity, a second order differential equation is used to demonstrate the method known as Variation of Parameters. The method shown can be generalized and applied to higher order differential equations. It turns out that in order to find a particular solution for (3)

$$a_2 y'' + a_1 y' + a_0 y = Q(x) \tag{3}$$

It is necessary to find two linearly independent solutions for the homogenous equation (4):

$$a_2 y'' + a_1 y' + a_0 y = 0 \tag{4}$$

The two linearly independent solutions will be:

$$y_c = c_1 y_1 + c_2 y_2 \tag{5}$$

(6) provides the definition of the Wronskian (determinant) which can be used to determine linear independence. If $\mathcal{W} \neq 0$, then the solutions are linearly independent.

$$\mathcal{W} = \begin{matrix} y_1 & y_2 \\ y_1' & y_2' \end{matrix} \tag{6}$$

The particular solution will look like (7)

$$y_p(x) = u_1(x) \times y_1(x) \ + \ u_2(x) \times y_2(x) \tag{7}$$

Where $u_1(x)$ and $u_2(x)$ are unknown functions and $y_1(x)$, $y_2(x)$ are the two linearly independent solutions previously mentioned. The general solution (8) is a combination of (5) and (7).

$$y(x) = y_c(x) + y_p(x) \tag{8}$$

Approach

The **first step** is to find (5) for the given second order differential equation. This requires the following substeps

(a) translate the differential equation into the correct characteristic form.

(b) determine the roots of the characteristic equation.

(c) derive two linearly independent solutions based on the type of roots found for the characteristic equation (real distinct, real repeated, or imaginary).

(d) optionally use (6) to verify that the solutions y_1 and y_2 are linearly independent.

Using (9) as an example, the substeps mentioned are shown in (10a), (10b), and (11). The Wronskian is not calculated.

$$y'' + 2\,y' + y = x^2\,e^{-x} \tag{9}$$

$$m^2 + 2\,m + 1 = 0 \tag{10a}$$

$$(m+1)\,(m+1) = 0 \tag{10b}$$

$$y_c = c_1\,e^{-x} + c_2\,x\mathrm{e}^{-x} \tag{11}$$

The **second step** is to calculate u_1'. This requires the following sub-steps

(a) calculate y_1' and y_2', the first derivatives of y_1 and y_2

(b) use (12) which requires evaluating a fraction consisting of two determinants in both the numerator and denominator.

$$u_1' = \frac{\left(\begin{vmatrix} 0 & y_2 \\ \frac{Q(x)}{a_2} & y_2' \end{vmatrix}\right)}{\left(\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}\right)} \tag{12}$$

Referring back to (11), $y_1' = -e^{-x}$ and $y_2' = -x\mathrm{e}^{-x} + e^{-x}$. The simplified answer for u_1' is shown in (13c).

$$u_1' = \frac{\left(\begin{vmatrix} 0 & x\mathrm{e}^{-x} \\ x^2\,e^{-x} & -x\mathrm{e}^{-x} + e^{-x} \end{vmatrix}\right)}{\left(\begin{vmatrix} e^{-x} & x\mathrm{e}^{-x} \\ -e^{-x} & -x\mathrm{e}^{-x} + e^{-x} \end{vmatrix}\right)} \tag{13a}$$

$$u_1' = \frac{\left(-x^3\right)e^{-2\,x}}{\left(e^{-2\,x}\right)} \tag{13b}$$

$$u_1' = \left(-x^3\right) \tag{13c}$$

The **third step** is to calculate u_2'. An inspection of (14) shows that the denominator is the same as (12). It is only necessary to calculate the determinant found in the numerator, re-use the denominator value from (12) and then simplify the fractional expression. This is demonstrated in steps (15a), (15b), and (15c).

$$u_2' = \frac{\left(\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{Q(x)}{a_2} \end{vmatrix}\right)}{\left(\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}\right)} \tag{14}$$

$$u_2' == \frac{\left(\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & x^2\, e^{-x} \end{vmatrix}\right)}{\left(\begin{vmatrix} e^{-x} & x\mathrm{e}^{-x} \\ -e^{-x} & -x\mathrm{e}^{-x} + e^{-x} \end{vmatrix}\right)} \tag{15a}$$

$$u_2' = \frac{\left(x^2\right) e^{-2\,x}}{\left(e^{-2\,x}\right)} \tag{15b}$$

$$u_2' = \left(x^2\right) \tag{15c}$$

The **fourth step** is to find $u_1(x)$ which requires integrating u_1' with respect to x. You can safely ignore the constant of integration. **Note: Which integration technique to use depends on the problem, I cannot tell you beforehand if you need to use integration by parts, trigonometric substitution, partial fractions approach, etc.**

$$u_1 = \int \left(-x^3\right) d\,x \tag{16a}$$

$$u_1 = \frac{-x^4}{4} \tag{16b}$$

The **fifth step** is to find $u_2(x)$ which requires integrating u_2' with respect to x. You can safely ignore the constant of integration.

$$u_2 = \int \left(x^2\right) d\,x \tag{17a}$$

$$u_2 = \frac{x^3}{3} \tag{17b}$$

The **sixth step** is to perform the arithmetic shown in (7). The result is the particular solution.

$$y_p(x) = \left(\frac{x^4}{4}\right) e^{-x} + \left(\frac{x^3}{3}\right) x\mathrm{e}^{-x} \tag{18a}$$

$$y_p(x) = \left(\frac{x^4}{12}\right) e^{-x} \tag{18b}$$

The **seventh step** is to perform the arithmetic shown (8). The result is the general solution.

$$y(x) = \left(\frac{x^4}{12}\right) e^{-x} + c_1\, e^{-x} + c_2\, x\mathrm{e}^{-x} \tag{19}$$